Localization in disordered superconducting wires with broken spin-rotation symmetry

Ilya A. Gruzberg[1], N. Read[2], and Smitha Vishveshwara[3]
[1] *The James Franck Institute, The University of Chicago, 5640 S. Ellis Avenue, Chicago, IL 60637*
[2] *Department of Physics, Yale University, P.O. Box 208120, New Haven, CT 06520-8120*
[3] *Department of Physics, University of Illinois at Urbana-Champaign, Urbana, IL 61801*
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Localization and delocalization of non-interacting quasiparticle states in a superconducting wire are reconsidered, for the cases in which spin-rotation symmetry is absent, and time-reversal symmetry is either broken or unbroken; these are referred to as symmetry classes BD and DIII, respectively. We show that, if a continuum limit is taken to obtain a Fokker-Planck (FP) equation for the transfer matrix, as in some previous work, then when there are more than two scattering channels, all terms that break a certain symmetry are lost. It was already known that the resulting FP equation exhibits critical behavior. The additional symmetry is not required by the definition of the symmetry classes; terms that break it arise from non-Gaussian probability distributions, and may be kept in a generalized FP equation. We show that they lead to localization in a long wire. When the wire has more than two scattering channels, these terms are irrelevant at the short distance (diffusive or ballistic) fixed point, but as they are relevant at the long-distance critical fixed point, they are termed *dangerously irrelevant*. We confirm the results in a supersymmetry approach for class BD, where the additional terms correspond to jumps between the two components of the sigma model target space. We consider the effect of random π fluxes, which prevent the system localizing. We show that in one dimension the transitions in these two symmetry classes, and also those in the three chiral symmetry classes, all lie in the same universality class.

I. INTRODUCTION

Localization of single particle states in disordered fermionic systems has been a key concept underlying our understanding of disordered normal metals. The nature of the states depends crucially on the presence or absence of time reversal (TR) and spin rotation (SR) symmetries. Based on these considerations, three distinct Wigner-Dyson symmetry classes were identified a long time ago — orthogonal, unitary and symplectic (the names come from the random matrix theory of Wigner and Dyson[1]).

Recently, it has been realized that localization ideas are useful in describing a wider class of systems including superconductors and superfluids, various quantum Hall states and random-bond Ising models. A crucial part of the emerging picture is again the role played by symmetries of the disordered single-particle Hamiltonians describing these systems. In addition to the TR and SR symmetries, these new systems may possess additional symmetries that have to be taken into account. One such category is comprised of systems defined on bipartite lattices with the sublattice (also called chiral) symmetry. Another category, on which this paper is focussed, includes Bogoliubov-de Gennes (BdG) Hamiltonians which describe quasiparticles in disordered superconductors at a mean field level. Although particle number is not conserved in the mean field description, the problem of finding the quasiparticle spectrum can be transformed to a particle-number conserving form, so that it reduces to solving a single-particle Hamiltonian that has certain symmetries. The spectra of Hamiltonians in all of these classes have an exact reflection symmetry around zero energy, and eigenstates near zero energy may have localization properties different from those of eigenstates in other parts of the spectra.

A symmetry classification of disordered single-particle Hamiltonians has been established by Altland and Zirnbauer (AZ)[2,3]. The symmetry class and the dimensionality of a particular disordered system in principle determine which phases (localized or extended) and phase transitions may occur in its phase diagram. After ensembles of random matrices, which are the zero-dimensional cases, the next simplest situation arises in one dimension (1D), where various analytical methods are available. In the standard Wigner-Dyson classes all the single-particle states are localized in 1D and this leads to an exponential decrease of the conductance of a 1D wire with its length, that is, the so-called strong localization.

Brouwer *et al.* in Ref. 4 studied localization in 1D (superconducting wires) using the Fokker-Planck (FP) scattering approach[5,6]. They considered all four BdG symmetry classes (denoted C, CI, D, and DIII in Ref. 2), and obtained the corresponding four FP equations describing transport at zero energy. While for the two classes C and CI that have unbroken SR invariance the quasiparticle (thermal) conductance $\mathcal{G}$ decays exponentially with the length L of the wire for large L, the situation was found to be quite different in classes D and DIII in which the SR invariance is absent. In the latter, the mean $\langle\mathcal{G}\rangle$ decays only algebraically to zero for large L and $\ln\mathcal{G}$ is not self-averaging, indicating a very broad distribution of the conductance and the absence of the exponential localization of the quasiparticle states. This behavior was termed critical in Ref. 4.

This surprising result was questioned by Motrunich *et al.* in Ref. 7. In this paper some models in symmetry classes D and DIII were studied using real-space renormalization group methods, and it was argued that, for a generic distribution of disorder, at zero energy there

are localized phases separated by a phase boundary at which the localization length diverges. It was found that close to the critical point, the low-energy density of states (DOS) per unit length, $\nu(\epsilon)$, has a divergent power-law behavior with a non-universal varying exponent:

$$\nu(\epsilon) \propto \epsilon^{-\delta}, \tag{1}$$

as $\epsilon \to 0^+$, with $\delta < 1$. At the critical point, one has $\delta = 1$, and there are logarithmic corrections of the Dyson form[8]:

$$\nu(\epsilon) \propto \epsilon^{-1} |\ln \epsilon|^{-3}. \tag{2}$$

In addition, as parameters vary, it is possible for the phase boundaries to collide, in which case there is a multicritical point with different critical properties.

In Ref. 9, the FP scattering approach was used to study the DOS in disordered wires in all symmetry classes. Perhaps unsurprisingly, for the wires in classes D and DIII the result had a singularity of the critical form in Eq. (2). In Ref. 10, Brouwer *et al.* tried to account for the discrepancies between the results of the two groups in terms of a crossover from the unitary class to class D within a particular model. We believe that this account leaves the role of symmetries and universality in the problem unclear, and that our work described here greatly clarifies these questions.

In this paper we reconsider localization in superconducting wires with broken spin-rotation symmetry, in classes BD and DIII in the AZ classification [we prefer the term BD in place of class D for more than zero dimensions, as the distinction between odd (B) and even (D) size Hamiltonian matrices disappears for infinite matrices], for a finite number N of propagating modes in the wires, and resolve the major issues. The central point concerns symmetries of the problem. The most popular type of models for disordered wires are those in which propagation along the wire is continuous, and this applies in particular for weak disorder. We show that for the cases of classes BD and DIII, there are certain symmetry operations on the spaces of transfer matrices (or on the probability distributions for the disorder) which have been overlooked previously. The conditions that define classes BD and DIII do *not* impose that the distribution be invariant under these operations. However, when one takes the standard limit that results in the FP equation, the leading symmetry-breaking term is a differential operator of order N, and terms of order greater than two disappear in the limit (these are related to deviations of the disorder distribution from Gaussian). That is, *the FP equations for classes BD and DIII studied by Brouwer et al[4,9] possess more symmetry than the general class of distributions does.* For $N = 1$, the leading symmetry-breaking term is a drift (first-order) term which produces localization if it is present. We show that for a long wire, the N channel case reduces to the $N = 1$ situation, and the leading symmetry-breaking term reduces to the drift term. Then generically, *breaking the additional symmetry produces localization.* In a generalization of the FP equation that keeps higher-order operators, operators of order higher than 2 would be irrelevant at short distance [in renormalization group (RG) language]. Since they are nonetheless relevant at large distances (they take the system off criticality), they cannot be dropped, and so they are analogous to "dangerously irrelevant" operators in critical phenomena. In practice, to represent a system that will have both a finite number N of channels, and a finite mean free path as an ultraviolet cutoff, the FP limit is not taken, and symmetry-breaking terms must be retained. The behavior near the critical point is universal, and is governed by the $N = 1$ case. The results agree with those of Motrunich *et al*[7].

In addition to demonstrating these points with the FP approach, we also give the supersymmetric representation of the wire for class BD, and show how the phenomena appear there. The wire is represented by a finite-N analog of the supersymmetric nonlinear sigma model (in 1D), which is known to have a target space that consists of two connected components in the cases of classes BD and DIII[11]. We show that the terms in the distribution of disorder that break the additional symmetry produce a jump from one component to the other whenever they occur in the expansion of a disorder average. These are domain walls in the sigma model; topologically, in 1D these domain walls are simply points. The supersymmetry approach is complementary to the FP equation; it produces results for the moments of Green's functions, and is used here to study the DOS.

For the more general situation in which the transfer matrix does not have to evolve continuously, as in a lattice (or network) model, it is possible for an amplitude to pick up a factor of -1 during a time step, which would correspond to encircling a flux of π in a lattice model; such an effect is consistent with the conditions that define classes BD and DIII. It was already shown[12] in the case of class BD with $N = 1$ that if such events are distributed independently and uniformly along the wire, the drift term in the FP equation is effectively suppressed, and the system is again critical. We show here that this also occurs for the case of general N, and then on scales larger than the separation of these events, the N-channel problem effectively maps onto the analysis as given in Ref. 4, and is always critical even when the symmetry-breaking terms are present. In the supersymmetry approach, domain walls are suppressed in this situation. Thus domain walls are essential to produce localization for all N, as is found also in two dimensions for class BD[11,12].

It is striking that the universal results for the symmetry classes BD and DIII turn out to be the same, and also are the same as those for the chiral classes. We show that this "super-universality" can be understood, as the $N = 1$ channel cases all possess hopping Hamiltonian models that can be mapped onto each other exactly.

The rest of the paper is structured as follows. In Section II we provide a brief account of the scattering approach to coherent transport in wires, stressing its symmetry aspects, and including the role of non-Gaussian

distributions, especially in the case of classes BD and DIII. In Section III we review the standard FP equation resulting from this approach, and its solution. We generalize the FP equation to include the leading terms that break the extra symmetries. We simplify the resulting generalized FP equation in the asymptotic regime of long wires and solve it, obtaining the scaling function of the logarithm of the conductance near the critical point, and exhibiting two localization lengths, the typical localization length for the typical values (and moments) of the logarithm of the conductance, and the mean localization length for the moments of the conductance itself. In Section IV, we describe aspects of the supersymmetry method for wires in the BD class. This gives us the scaling of the mean localization length and the mean density of states near the critical point. In Section V we explain what occurs in more general models in which the propagation along the wire is discontinuous, allowing fluxes of π to be randomly inserted. The mappings that establish super-universality are given in Section VI. Some final discussion of universality is given in the conclusion, Section VII. Finally, many technical details are presented in Appendices.

II. SYMMETRY ASPECTS OF THE TRANSFER MATRIX

In this section we consider the symmetry aspects within the transfer matrix formulation, which will be used in both the FP and supersymmetry approaches, and show that for classes BD and DIII with more than two scattering channels there are additional symmetries present in a Gaussian distribution of disorder that are not generically there. In order to make the key points in our analysis, it will be useful to reconsider briefly the basics of the transfer matrix approach to one-dimensional (1D) quantum transport (for more details, see Ref. 13).

It is convenient to pass from the single-particle Hamiltonian $\mathcal{H}$ to a scattering description, by considering waves of a given energy that enter the system at either end, are scattered and then emerge at the same or the opposite end. We will assume that only a finite number of "channels" are relevant at the energy considered. For the non-standard ensembles, the energy chosen has the special value $\epsilon = 0$ (this corresponds to calculating the transport properties of the ground state of the superconductor). The scattering can then be represented by an S-matrix that maps incoming to outgoing waves, or, better, by a transfer matrix M' that maps the in- and outgoing waves at one end (say, the right) to those at the other (the left); both of these matrices are $2N' \times 2N'$. The advantage of the transfer matrix is that 1D systems can be composed end-to-end by joining together short elements, and multiplying the respective transfer matrices, see Fig. 1. Taking the latter as random, identically-distributed, statistically-independent matrices, leads to an equation that describes the evolution of the probability distribution for the transfer matrix with increasing length of the system.

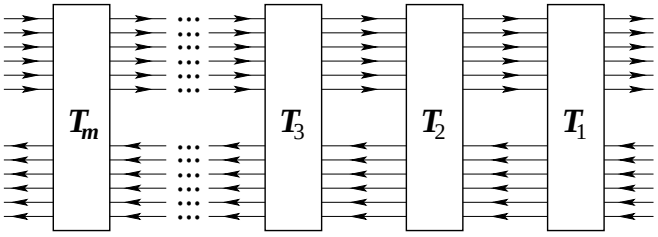

FIG. 1: The transfer matrix for the wire is composed of m steps, with independently-distributed increments T_j, $j = 1, \ldots, m$.

The transfer matrix is related to the Landauer conductance of the system. The Landauer-Buttiker formula (see Ref. 14 for history) for the two-probe (electrical) conductance reads

$$\mathcal{G} = \mathrm{tr}\ \boldsymbol{t}\boldsymbol{t}^\dagger, \tag{3}$$

where $\boldsymbol{t}$ is the $N' \times N'$ transmission matrix for waves entering at one end to exit at the other. In superconductors, charge (or quasiparticle number) is not a good quantum number at mean field level, but the Landauer conductance is still relevant for the thermal conductance, since heat is transported by quasiparticles: the thermal conductance is given by $\frac{\pi^2}{6} k_B^2 T \mathcal{G}$. The conductance can be rewritten in terms of the transfer matrix as

$$\mathcal{G} = 2\,\mathrm{tr}\left(2 + M'M'^\dagger + (M'M'^\dagger)^{-1}\right)^{-1}. \tag{4}$$

Thus it is a trace of a function only of $M'M'^\dagger$ (or of $M'^\dagger M'$).

The mathematical properties of M' are as follows. Because of conservation of probability flux ("current") in time-independent scattering, the S-matrix is unitary, which implies that the transfer matrix is pseudounitary:

$$M'^\dagger \Lambda M' = \Lambda, \tag{5}$$

where $\Lambda = \begin{pmatrix} I_{N'} & 0 \\ 0 & -I_{N'} \end{pmatrix}$. [The first N' indices refer to waves entering at the right (or leaving at the left), the second to those leaving at the right (or entering at the left).] Thus M' is a member of the noncompact Lie group $\mathrm{U}(N', N')$. For the unitary ensemble, this is the only condition, but for the other ensembles[2], additional conditions are imposed on M'. They result from the symmetries of the Hamiltonian $\mathcal{H}$ that define the symmetry class, and have the effect of restricting M' to a noncompact subgroup G of $\mathrm{U}(N', N')$. In some cases, the representation of G in $\mathrm{U}(N', N')$ may be reducible[4]. That is, there exists a unitary transformation to a basis in which the full $2N' \times 2N'$ transfer matrix M' becomes block diagonal, each of the d blocks being (for symmetry reasons) the same $2N \times 2N$ transfer matrix M (or in some cases, its adjoint, or transpose), so $N' = dN$, where

d depends on the symmetry class. We will call N (rather than dN) the number of channels, as it is the number of independent variables propagating in each direction.

In this section, and in the remainder of this paper except Section V, we aim to model the situation in which the propagation along the wire is continuous. For calculational convenience, we nonetheless represent this by discrete steps, as multiplication of transfer matrices for each step. The underlying continuity implies that after each step, M' lies in G and can be continuously connected to the identity, and thus G must be a connected group. For class BD, $M' = M$ can be taken as real, so G is $\mathrm{SO}_0(N, N)$ with $N' = N$ (as always, the prefix S means unit determinant, while the subscript 0 denotes the component containing the identity). For class DIII, G is isomorphic to $\mathrm{SO}(2N, \mathbb{C})$ ($N' = 2N$). *For the purposes of most of this paper, these statements can be taken as the definitions of these symmetry classes.* Tight-binding models of superconductors, though maybe not strictly continuous, lead to these systems with N even in both symmetry classes: Pairing of spinless fermions (without time-reversal symmetry) gives class BD, while pairing of spin-1/2 fermions with time-reversal symmetry, but with broken spin-rotation symmetry, gives class DIII. Pairing of fermions with spin 1/2, but with spin-rotation and time-reversal symmetries broken, gives class BD with N divisible by 4. Our conventions differ from those of Ref. 4, where their N is half ours for class DIII, and one-quarter of ours for class BD because they considered the case of spin-1/2 fermions. Similarly, some of the models studied in Ref. 7 are in class BD with $N = 2$ or 4, and others are in class DIII with $N = 2$. Our analysis applies to all positive integer values of N in both classes, and one odd value, $N = 1$, will play a special role in the analysis.

Mathematically, class DIII Hamiltonians for a 1D wire can also lead to transfer matrices in $\mathrm{SO}(2N+1, \mathbb{C})$. Here we would have $N' = 2N+1$ instead of $N' = 2N$. The behavior of these systems is different from that of those we analyze here, and somewhat similar to the odd number of channels case of the symplectic (AII) symmetry class[15]; namely the conductance of a long wire approaches $\mathcal{G} = 1$, an effect that was found in Ref. 16. These "$N + 1/2$-channel" class DIII systems can be realized as edge states of certain class DIII superconductors in two space dimensions. We do not consider these cases further in this paper.

The formula (4) for the conductance becomes, in terms of M,

$$\mathcal{G} = 2d \,\mathrm{tr}\left(2 + MM^\dagger + (MM^\dagger)^{-1}\right)^{-1}, \tag{6}$$

and is invariant under multiplication of M on the left *or* right by unitary matrices. Those that lie in G represent some redundant aspect of the evolution of M, in the sense that some scattering does not affect the conductance. We can find a maximal unitary (hence compact) subgroup K of G, such that $\mathcal{G}$ is invariant under $M \to k_1 M k_2, \quad k_1, k_2 \in K$. This suggests parameterizing M so as to factor off this subgroup. For example, for class BD, $K = \mathrm{SO}(N) \times \mathrm{SO}(N)$, and for DIII, $K = \mathrm{SO}(2N)$. Then any transfer matrix M for class BD can be written schematically as $M = k_1 a k_2$, or explicitly

$$M = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \begin{pmatrix} \cosh X & \sinh X \\ \sinh X & \cosh X \end{pmatrix} \begin{pmatrix} V_3 & 0 \\ 0 & V_4 \end{pmatrix}, \tag{7}$$

where $V_1, \ldots, V_4 \in \mathrm{SO}(N)$. For class DIII, we have instead

$$M = V_1 \begin{pmatrix} \cosh X & i \sinh X \\ -i \sinh X & \cosh X \end{pmatrix} V_2, \tag{8}$$

and V_1, V_2 are in $\mathrm{SO}(2N)$. Thus the group can be decomposed as $G = KAK$ (the Cartan decomposition of G), where A stands for the matrices containing X in the above expressions. In both cases, $X = \mathrm{diag}(x_1, \ldots, x_N)$ is real and diagonal. These parametrizations are also called radial or polar decompositions. The real parameters x_i are called radial coordinates, and the matrices $V_1, \ldots, V_4$, (resp., V_1, V_2) represent angular coordinates. Similar decompositions exist for other groups. A familiar example of the Cartan decomposition in a compact group is to take $G = \mathrm{SU}(2)$, $K = \mathrm{U}(1)$ (generated by the σ_z Pauli matrix), and then the three parameters in the decomposition are exactly the Euler angles that describe a rotation.

If we identify elements of G modulo right multiplication by elements of K, we obtain the coset space G/K. If we also identify points of this space modulo the action of K given by left multiplication, then we obtain the double coset space $K\backslash G/K$. The x_i parametrize this double coset space $K\backslash G/K$, and the Landauer conductance $\mathcal{G}$ is a function only of these radial coordinates:

$$\mathcal{G} = d \sum_{i=1}^{N} \frac{1}{\cosh^2 x_i}. \tag{9}$$

To ensure that each point of $K\backslash G/K$ is labeled by a single set of x_i, there should be restrictions on the possible values of x_i. We notice that it is possible to map the set A into itself by conjugation by certain elements of K, $a \to kak^{-1}$. Since left and right multiplication by K is to be divided out, these give the same point in the double coset space. The group of such operations is called the Weyl group of the Cartan decomposition in question. By inspecting K for the cases of class BD and DIII, we see that it is possible in this way to permute the x_is, and to reverse the sign of an *even* number of x_is [because for class BD, K involves $\mathrm{SO}(N)$, not $\mathrm{O}(N)$, and similarly for DIII]. Consequently, a unique parametrization for $K\backslash G/K$ is obtained by using the following domain (known as the Weyl chamber) for x_i:

$$C = \{|x_1| < x_2 < \ldots < x_N\}, \tag{10}$$

where x_1 can be either positive or negative.

We want to note here that all the previous treatments of classes BD and DIII in 1D based on the scattering

approach, including Refs. 4,9,10, incorrectly assumed a different range for x_i:

$$0 < x_1 < x_2 < \ldots < x_N. \tag{11}$$

This range is the one appropriate for all the standard (Wigner-Dyson) symmetry classes, and for the BdG classes C and CI in the AZ classification. For the chiral classes, the range is

$$x_1 < x_2 < \ldots < x_N. \tag{12}$$

The property that at least one of the radial coordinates (x_1) can take negative values is shared by the classes BD, DIII, and the chiral ones. Ultimately, as we show below, this allows phase transitions to occur in these classes. Within the FP approach, a transition at which the localization length diverges occurs when the mean of an x_i (in the limit of a long wire) passes through zero. There is one such transition in classes BD and DIII, but N in the chiral classes. (Related distinctions between the two phases, implying that there must be a transition, appeared in Refs. 7,17.) The distinction in the ranges for x_i is crucial to our symmetry analysis also, as we will see that the additional symmetry mentioned in the introduction acts on the double coset space by reversing the sign of x_1.

We next consider M as a product of $m = L/a$ elements of G:

$$M_m = T_m T_{m-1} \ldots T_1. \tag{13}$$

This discrete form could represent a discrete 1D model with lattice spacing a, or could be used to take a continuum limit $a \to 0$ with L fixed. In order to increase the length from m to $m+1$, we multiply M_m on the left by T_{m+1}. In general, we take T_j to be a random variable, in which case we obtain a stochastic process. If we assume that T_js at different j are statistically independent, and that all T_j are identically distributed, then we obtain a process whose dynamics is "time"-independent, where by "time" we mean position j along the wire (this terminology will be used consistently throughout this paper). Also, the time-evolution process is invariant under right multiplication by any element of the full group G, in the sense that $(T_{m+1}M)g = T_{m+1}(Mg)$ for $g \in G$, and in particular under right multiplication by any element of K. Thus it can be viewed as a stochastic process on the coset space G/K (and this is the point of view most often taken in the literature, see Refs. 18, 19). But it is not in general invariant under left multiplication even by elements of K, unless the distribution of T_j is invariant under $T_j \to kT_jk^{-1}$ for all $k \in K$ and all j. Unless this assumption is made, we do not obtain a well-defined stationary stochastic process on the double coset space $K\backslash G/K$, because the effect of left-multiplication by T_{m+1} on a double coset KgK (where each K stands for all members of K) depends on which element of the K is chosen at the left.

It is generally believed, however, that even in a generic situation, due to the compactness of the subgroup K, the diffusion in the angular variables will lead eventually (for long wires) to a uniform distribution on K (an ergodicity assumption). Therefore, one can use K-invariant distributions for T_j to analyze localization properties in long wires, without loss of generality. Together with the K-invariant initial condition $M = I$ for zero length, we can view the process as taking place on the double coset $K\backslash G/K$. All information necessary to obtain the probability distribution for $\mathcal{G}$ is contained in the joint distribution function of the radial coordinates $P(x_1, ...x_N; L) \equiv P(x; L)$. It depends on the length of the wire L, and our goal is to find the equation governing the evolution of $P(x; L)$ when L increases.

Each T_{m+1} is the transfer matrix of a thin slice, and as such is best parameterized as $T_{m+1} = \exp A$. Here A belongs to the Lie algebra of G, Lie(G), and the probability distribution for A is assumed to be concentrated near zero since the slice is thin. Also, due to the invariance of the distribution, A may be assumed to be in the subspace $\mathcal{P}$ of Lie(G) which appears in the infinitesimal version of the Cartan decomposition Lie(G) = Lie(K)+$\mathcal{P}$. (An element of $\mathcal{P}$ can be diagonalized using elements of Lie(K), as we used earlier, but we do not do this here.) Let us illustrate this point for class BD. In this case Lie(G) = so(N, N), Lie(K) = so(N) + so(N), and the subspace $\mathcal{P}$ consists of matrices of the form

$$A = \begin{pmatrix} 0 & \theta \\ \theta^T & 0 \end{pmatrix}, \tag{14}$$

where θ is a real $N \times N$ matrix. This form for A corresponds to motion in G/K starting from the point M_mK, and relative to such an origin the matrices θ parametrize G/K. For class DIII, $A = \theta$ is a purely imaginary antisymmetric $2N \times 2N$ matrix.

It will be useful to examine the case $N = 1$ for class BD explicitly as motivation for the symmetry arguments that follow, and because it plays a central role in the analysis even for general N. In this case, $G = \mathrm{SO}_0(1,1)$, K is trivial, and M is described by a single real number x_1, which is easily seen to be $x_1 = \sum_j \theta_j$. Thus the evolution process is a random walk on the real line parametrized by x_1 or θ. Clearly, if this random walk has net drift of either sign (i.e. the mean of θ_j is nonzero), then x_1 goes to plus or minus infinity with high probability, and the conductance becomes exponentially small, indicating localization. If the mean of θ_j is zero, the distribution of x_1 will have width of order $\sqrt{L}$ (we assume that the distribution for θ_j has finite variance). In this case, the behavior is of the same form as the critical behavior for class BD, mentioned in the introduction[4]. This behavior will be considered in more detail below; here we wish only to draw attention to the central feature, which is that there is an obvious symmetry operation on $\mathrm{SO}_0(1,1)$, given by reversing the sign of x_1, and that the drift breaks this symmetry. This symmetry is a discrete operation, and cannot be obtained by using the left and right action of

$\mathrm{SO}_0(1,1)$ on itself. There is a similar analysis for $N=1$ in class DIII: $\mathrm{SO}(2,\mathbb{C})$ can be represented by the group of nonzero complex numbers z under multiplication (in fact, the eigenvalues of M are z, z^{-1}), $K=\mathrm{SO}(2)$ is represented by the complex numbers of modulus 1, and the additional symmetry is given by $z\to z^{-1}$. Localization occurs if z drifts to either 0 or ∞; here $x_1=\ln|z|$.

As an aside, we see from this analysis that the double coset spaces for classes BD and DIII for $N=1$ are the same. In fact, the symmetric spaces G/K for the two cases are also the same space: both are simply the real line, with translation symmetry along the line (in suitable variables). For DIII this relies on the fact that $\mathrm{SO}(2,\mathbb{C})$ is Abelian, and so $K\backslash G/K$ and G/K are the same. Moreover, there is a similar symmetry discussion for all three chiral symmetry classes (orthogonal, unitary, and symplectic, or classes BDI, AIII, and CII respectively in the AZ classification), and for these the double coset and symmetric spaces are also the same space (the real line) for $N=1$. Therefore, at least for properties that can be expressed in terms of the transfer matrix, the critical behavior that occurs in each of these cases is the same. While this observation may seem trivial at this stage, it will become much more significant once we see that the $N=1$ cases control the long-distance behavior of the critical systems at $N>1$ also. Such equivalences, which rely on well-known isomorphisms of Lie groups of small rank, also occur in some other cases of the transfer matrix approach to disordered wires. In Section VI below, we establish mappings between the $N=1$ cases in the above five classes directly, in terms of hopping Hamiltonians.

It is natural to ask whether there are any analogous additional symmetries for classes BD and DIII in the cases with $N>1$. It turns out that there are. In both cases, there is a larger group that contains G as a connected subgroup: these are $\mathrm{O}(N,N)$ and $\mathrm{O}(2N,\mathbb{C})$. $\mathrm{O}(N,N)$ has four connected components, including two in which the elements have determinant -1, and the part $\mathrm{SO}(N,N)$ with determinant $+1$ has two connected components, one of which is $\mathrm{SO}_0(N,N)$. $\mathrm{O}(2N,\mathbb{C})$ has just two components, with determinants ±1 respectively. Certain subgroups of these act on our groups G, mapping it into itself. In particular, as we wish to leave the identity element fixed, there is the action by conjugation, $g\to hgh^{-1}$, where $g\in G$, and h is in the larger group. Examples for h of the desired form can in fact be found by choosing h in $\mathrm{O}(N)\times\mathrm{O}(N)$ or $\mathrm{O}(2N)$, for class BD or DIII respectively; in both cases, such an h has the desired effect if it has negative determinant. Then the product of any two such h lies in the subgroup $\mathrm{S}[\mathrm{O}(N)\times\mathrm{O}(N)]$ or $\mathrm{SO}(2N)$, respectively. (In the special case of class BD with N odd, we can take h to be Λ, which commutes with any element of K.) For the action on the double coset space, we can compare this discussion with that of the Weyl group, and we see that we now have the set of all permutations and sign reversals of the x_i, so that in terms of the Weyl chamber C, the additional symmetry is simply to reverse the sign of x_1. Mathematically, what we have described is an outer automorphism (or involution) of the reduced root system of G/K (outer means it is not connected to the identity). As these are of type D_N in both cases, and the outer automorphisms are known to be related to the existence of symmetries of the Dynkin diagram, we do find such a symmetry. [There are similar additional symmetries for each of the chiral symmetry classes. In these cases, the operation that sends x_i to $-x_i$ for *all* i is such a symmetry, an outer automorphism of the reduced root systems, which are all of type A_{N-1} for these classes. However, these symmetries cannot be written as conjugation by elements of a covering group of G, unlike the BdG cases above.]

We saw in the examples with $N=1$ that if the distribution of the random transfer matrices T_j for each slice is invariant under the additional symmetry (as well as under conjugation by K), then localization would be prevented. As localization requires that all x_i including x_1 go off to $\pm\infty$, it is now clear that the same will be true for $N>1$. On the other hand, if the distribution is perturbed so as to break the symmetry, then localization may set in. [For the chiral classes, there is a similar effect: if the symmetry under $x_i\to-x_i$ for all i is preserved, then localization does not occur (the system is critical) if and only if N is odd[20].] We must now determine the form of probability distributions that satisfy the symmetry requirements.

A popular, and usually effective, choice is to take the distribution for the matrix θ in each time slice to be Gaussian,

$$P[A]\propto\exp\left(-\frac{C_2\ell}{a}\mathrm{tr}\,A^2\right), \tag{15}$$

where C_2 is a constant that can be chosen to simplify some later equations, and ℓ is the "mean free path". For a Gaussian, only the second cumulant is nonzero. For class BD, it takes the form (for $N\neq2$)

$$[\theta_{a_1b_1}\theta_{a_2b_2}]_c=c_2\delta_{a_1a_2}\delta_{b_1b_2}, \tag{16}$$

where $c_2=a/4C_2\ell$. For a generic K-invariant distribution, higher cumulants are also nonzero, and should be expressed in terms of invariant tensors of K. Let us illustrate this for class BD. In this case, for an element k of $K=\mathrm{SO}(N)\times\mathrm{SO}(N)$ we write

$$k=\begin{pmatrix}k_1&0\\0&k_2\end{pmatrix}, \tag{17}$$

where $k_i\in\mathrm{SO}(N)$. Then as K acts on T_m by conjugation, $T_j\to kT_jk^{-1}$, this gives

$$\theta\to k_1\theta k_2^{-1}=k_1\theta k_2^T, \tag{18}$$

and we see that the second cumulant (16) is expressed in terms of $\mathrm{SO}(N)$-invariant Kronecker delta in the left and right indices of θ, so is invariant under $\mathrm{SO}(N)\times\mathrm{SO}(N)$. Moreover, it is invariant under $\mathrm{O}(N)\times\mathrm{O}(N)$. Invariant

tensors under O(N)×O(N), acting in this fashion on θ, all have a similar form involving Kronecker δs in the left and right indices. (The cumulants

$$[\theta_{a_1b_1}\theta_{a_2b_2}\dots\theta_{a_{2k}b_{2k}}]_c \tag{19}$$

must also be invariant under permutations of pairs a_i, b_i.) The number of such invariant expressions increases rapidly with the degree of the cumulant, and the $2k$th cumulant contains a distinct parameter for each one.

However, there is also another invariant tensor of SO(N), which is not invariant under O(N): the completely antisymmetric Levi-Civita tensor $\varepsilon_{a_1a_2\dots a_N}$. Hence, there can be a term in the Nth cumulant of the form

$$c'_N\varepsilon_{a_1a_2\dots a_N}\varepsilon_{b_1b_2\dots b_N}. \tag{20}$$

For even N, the Nth cumulant will contain contributions of both types. At higher degrees, there are also invariants that are products of both one or more Levi-Civita tensors and Kronecker δs. Under the additional symmetry, which corresponds to acting on θ with k_1 and k_2, where $\det k_1 \det k_2 = -1$, the Kronecker δ function is invariant, but the Levi-Civita tensor reverses sign. Thus the cumulant (20) is the one of lowest degree that (if c'_N is nonzero) breaks the additional symmetry, but not that under K.

For class DIII, the matrix θ is imaginary and antisymmetric. $K = \mathrm{SO}(2N)$ acts on it by conjugation. Consequently, there are K-invariant cumulants containing Kronecker δs in arbitrary pairs of indices (the lowest being $[\theta_{a_1a_2}\theta_{a_3a_4}]_c = -c_2\big(\delta_{a_1a_3}\delta_{a_2a_4} - \delta_{a_1a_4}\delta_{a_2a_3}\big)$ with $c_2 = a/4C_2\ell$), which are invariant under O($2N$), and the lowest-order form that is invariant under SO($2N$) but not under O($2N$) is now

$$c'_N\varepsilon_{a_1a_2\dots a_{2N}}, \tag{21}$$

which is again Nth order in θ.

Thus in both classes BD and DIII, for $N > 2$ the use of a Gaussian distribution loses the symmetry-breaking aspect of the generic distribution, and introduces an additional symmetry into the evolution of the probability distribution with the length of the wire. On the other hand, for $N = 1$, the cumulant that breaks the additional symmetry is degree one, and corresponds to the nonzero mean or drift for θ. For $N = 2$, the symmetry-breaking term is degree two, the same as that obtained from the invariant Gaussian. Both terms in the cumulant can be obtained by using a more general form of Gaussian distribution[9], of the forms

$$P[A] \propto \exp\left(-\frac{C_2\ell}{a}\mathrm{tr}\,A^2 - \frac{C'_2\ell}{a}\sum_{\substack{a_1,a_2\\b_1,b_2}}\varepsilon_{a_1a_2}\varepsilon_{b_1b_2}\theta_{a_1b_1}\theta_{a_2b_2}\right) \tag{22}$$

for class BD, and

$$P[A] \propto \exp\left(-\frac{C_2\ell}{a}\mathrm{tr}\,A^2 - \frac{C'_2\ell}{a}\sum_{a_1,\dots,a_4}\varepsilon_{a_1a_2a_3a_4}\theta_{a_1a_2}\theta_{a_3a_4}\right) \tag{23}$$

for class DIII. Thus also for $N = 2$, one might expect that localization can be induced by introducing the distributions with nonzero C'_2 (the corresponding effect on the DOS was observed in Ref. 9).

In the rest of this paper, we analyze explicitly the role of symmetry breaking by the probability distribution for T_j, within the FP approach, for all N, which means that in general the FP equation has to be generalized to incorporate non-Gaussian effects. We also perform an analysis (for class BD) within the supersymmetry approach.

III. GENERALIZED FOKKER-PLANCK EQUATION

In this section we first briefly review the main aspects of the continuum limit or FP approach for the BdG symmetry classes. Then for classes BD and DIII we argue that while, for well-behaved distributions, the cumulants that break the additional symmetry are lost in the usual continuum limit, in general for the analysis of the long time behavior they should be included as higher-order differential operators within a generalized FP equation. Then we show that at long times, where the problem reduces effectively to the $N = 1$ case, these terms reduce to the drift term and produce localization.

If one uses a Gaussian distribution of the form (15) independently for each T_j, then in the continuum limit $a \to 0$ one obtains a heat (diffusion) or FP equation on G, which can be reduced to one on the coset space G/K (the derivation is reviewed below for a more general distribution of T_j). In view of the invariance under the left action of K of both the stochastic process on G/K and the initial condition, the probability distribution on G/K depends only on the radial coordinates x_i, and the process can be viewed as one on the double coset space $K\backslash G/K$ for the joint distribution $P(x;L)$ of the x_i (using the initial condition $P(x;0) = \prod_i \delta(x_i)$). For the BdG classes, this FP equation takes the form (first studied in the physical context in Ref. 4):

$$\frac{\partial P}{\partial L} = \frac{1}{2\gamma\ell}\sum_{i=1}^{N}\frac{\partial}{\partial x_i}J\frac{\partial}{\partial x_i}J^{-1}P, \tag{24}$$

$$J(x) = \prod_{i=1}^{N}\sinh^{m_l}2x_i\prod_{i<j}^{N}\prod_{\pm}\sinh^{m_o}(x_j \pm x_i), \tag{25}$$

where m_l and m_o are non-negative integers (multiplicities of reduced roots) which depend on the symmetry of the problem ($m_o = 1$, $m_l = 0$ for class BD, and $m_o = 2$, $m_l = 0$ for class DIII), and $\gamma = m_o(N-1) + m_l + 1$.

[To obtain this form, we have now fixed the previously-introduced parameter C_2 in terms of γ, by $\gamma = 4C_2$.] The differential operator

$$D_2 \equiv \sum_{i=1}^{N} \frac{\partial}{\partial x_i} J \frac{\partial}{\partial x_i} J^{-1}, \tag{26}$$

which appears in the right-hand side of Eq. (24) is related to the part Δ_2 of the Laplace-Beltrami operator on the space G/K that is independent of the angular coordinates (for the information on analysis on symmetric spaces, see Refs. 21–24; relevant facts are also given in Appendix A):

$$D_2 = J \Delta_2 J^{-1}. \tag{27}$$

The factor of J arises because P is the probability density on $K\backslash G/K$, not on G/K. It is the Jacobian that appears in the G-invariant integration measure over G/K in terms of the radial coordinates x_i and angular coordinates in K. For classes BD and DIII, J and D_2 are invariant under $x_1 \to -x_1$.

Pertinent aspects of the solution of the Fokker-Planck equation are reviewed in Appendix B. In particular, the solution has a simple form for long wires when $L \gg \gamma\ell$:

$$P(x;L) \approx \Big(\frac{\gamma\ell}{2\pi L}\Big)^{N/2} \prod_{i=1}^{N} \exp\Big(-\frac{\gamma\ell}{2L}\big(x_i - L/\xi_i\big)^2\Big), \tag{28}$$

$$\xi_i = \gamma\ell\big(m_o(i-1) + m_l\big)^{-1}. \tag{29}$$

The asymptotic form (28) is a product of Gaussian factors, one for each radial coordinate. This implies that for large values of $L/\gamma\ell$, the radial coordinates x_i are with high probability well separated compared with their fluctuations. This behavior of x_i is stated precisely in the Oseledec theorem[25], which says that the x_i become statistically-independent and Gaussian-distributed with means and variances proportional to L:

$$\langle x_i \rangle = L/\xi_i, \qquad \text{var } x_i = L/\gamma\ell. \tag{30}$$

In these relations the quantities ξ_i are called localization lengths, and their inverses are the Lyapunov exponents.

Here we note that the easiest way to understand Eqs. (28–30) is to take the limit of large and widely separated x_i directly in the Fokker-Planck equation. As explained in Appendix A, in this limit the expression for the Jacobian $J(x)$ simplifies, so that $\partial \ln J/\partial x_i \to 2\big(m_o(i-1) + m_l\big)$, leading to the asymptotic form of the Fokker-Planck equation:

$$\frac{\partial P}{\partial L} = \frac{1}{2\gamma\ell} \sum_{i=1}^{N} \left(\frac{\partial^2 P}{\partial x_i^2} - 2\big(m_o(i-1) + m_l\big) \frac{\partial P}{\partial x_i} \right). \tag{31}$$

This is a simple diffusion equation with a constant diffusion coefficient and a drift, and its solution (using the same initial condition as before) is Eq. (28).

The Landauer conductance (9) of a long wire is dominated asymptotically by the smallest radial coordinate x_1:

$$\mathcal{G} \sim d/\cosh^2 x_1. \tag{32}$$

Thus, the leading behavior of both the mean and the typical values of $\mathcal{G}$ is determined by the longest of the localization lengths $\xi \equiv \xi_1 = \gamma\ell/m_l$, corresponding to the smallest Lyapunov exponent. In most symmetry classes $\xi_1 < \infty$, and the leading behavior of $\mathcal{G}$ is exponentially small, and this is referred to as strong or exponential localization.

A distinctive feature of the FP equation for classes BD and DIII, first investigated in the physical context in Ref. 4, is that the root systems of the corresponding symmetric spaces do not have long roots: $m_l = 0$. Consequently, there is no drift term for x_1 in Eq. (31), which results in x_1 performing a random walk about zero. (We now understand this behavior as a consequence of the additional symmetry of these two double coset spaces and of the FP equations (24) under $x_1 \to -x_1$.) Then the smallest Lyapunov exponent $1/\xi_1$ vanishes, leading to the absence of exponential localization in long wires. In this case, we should be a little more careful about the treatment of x_1 as L becomes large. We wish to assume that all $|x_i|$ are well-separated, so that $x_2, \ldots, x_N$ are large, but not necessarily that $|x_1|$ is large. For the cases of interest, with $m_l = 0$, we find that J is asymptotically independent of x_1, and that Eq. (31) is still valid in this regime (even at small x_1). The behavior of the conductance $\mathcal{G}$ in these cases is characterized by a very broad distribution, with an algebraic decay of the mean and the variance, while the variance of $\ln \mathcal{G}$ is of order L^4:

$$\langle \mathcal{G} \rangle \sim d\sqrt{\frac{2\gamma\ell}{\pi L}}, \qquad \text{var}\, \mathcal{G} \sim \frac{2}{3} d \langle \mathcal{G} \rangle,$$
$$\langle \ln \mathcal{G} \rangle \sim -4\sqrt{\frac{L}{2\pi\gamma\ell}}, \quad \text{var}\, \ln \mathcal{G} \sim \frac{4(\pi-2)L}{\pi\gamma\ell}. \tag{33}$$

Note that the mean conductance in this situation is "super-Ohmic", since Ohmic conductance decays as L^{-1}. The wire with this behavior of the conductance was called *critical* in Ref. 4.

The preceding FP analysis was for the $a \to 0$ limit of a Gaussian distribution invariant not only under K but under the additional symmetry. For general N, especially $N > 2$, we wish to include the effect of a non-Gaussian distribution for the T_j into the analysis, since for $N > 2$ these are the effects that break the larger symmetry down to K and may lead to localization. (The following analysis also includes the cases $N = 1$, 2, in which these effects occur already at the level of general K-invariant Gaussian distributions.) If we wish to cast this into the very convenient form of a differential equation in continuous time, then we must first consider the $a \to 0$ limit for the more general situation of an arbitrary distribution for each time step. (This may be a suitable place at which

to mention that the use of discrete but statistically independent increments T_j can be used to model not only processes that are actually discrete, but also those that are continuous but not δ-correlated in time, using a of order of the correlation time, provided the correlations are not long range. In practice, the correlation time might be of about the same order as the mean free path. As we will see, it is in these situations that the symmetry breaking effects for $N > 2$ can show up.)

In the standard derivation of a FP equation, there is some distribution for each (independent, identically-distributed) increment of the random variables being considered. In the following, we make the assumption that this distribution has finite moments of all degrees. The change in the probability density for the variables at time $m + 1$ from that at time m at any point is given by the master equation, and contains two terms, one for reduction of the probability by transitions to all other values, the other for the increase by transitions from other values (we give only a sketch of this derivation, as it can be found in many textbooks, and adapted to the present case without difficulty). Since the distribution of increments is narrow, the increments in the variables are typically small, and in the second term the probability distribution for time m can be Taylor expanded in the increments, about the values at time $m + 1$. The convolution integral over these increments then reduces in each term to a multiple partial derivative of the product of a moment of the distribution for the increment and the probability distribution. The zeroth-order such term cancels the first term in the master equation. The result, in which we have not yet taken any limit $a \to 0$, is a generalization of the FP equation which is a first-order finite difference equation in the time variable m, and all orders in partial derivatives in the variables. Such an expansion for a finite increment in the variables in terms of derivatives of the distribution is known as a Kramers-Moyal expansion, see, for example, Ref. 26.

In the present case, using a K-invariant distribution of the increments (described by the matrix A), each term is a differential operator on G/K that is invariant under the left action of the whole of G. Each operator of order k has a coefficient that is related to a kth moment of the distribution of increments. By approximating the time step of length a by a first derivative, we obtain a generalized form of FP equation for the probability distribution $P(\theta; L)$ on G/K, where the variables θ introduced earlier also serve as the coordinates on G/K, with $\theta = 0$ being the origin, which corresponds to the transfer matrix M being the identity in G. The G-invariant differential operators on symmetric spaces have been extensively investigated, see for example Ref. 22. Because sums, differences, and products of invariant operators are also invariant, the G-invariant operators form an algebra, which turns out to be commutative, and in fact all the operators can be obtained as linear combinations of products of a finite set of operators. For both classes BD and DIII, these operators can be labelled Δ_2, Δ_4, ..., Δ_{2N-2}, each of which has order given by its subscript, and Δ'_{p_N}, which is of order N. Thus the generalized FP equation can be written as

$$a\frac{\partial P(\theta;L)}{\partial L} = \sum_{n_1,\ldots,n_N} \frac{m_{n_1,\ldots,n_N}}{k_{\rm tot}!} \Delta'^{n_N}_{p_N} \left(\prod_{k=1}^{N-1} \Delta_{2k}^{n_k} \right) P(\theta;L), \tag{34}$$

where the coefficients $m_{n_1,\ldots,n_N}$ are essentially moments of degree $k_{\rm tot} = \sum_{k=1}^{N-1} 2kn_k + Nn_N$ of the increments θ, except for $m_{0,\ldots,0}$ which is zero due to probability conservation. [Strictly, these are the moments of θ minus its mean, except for the coefficient of the first-order operator, if any, which is the mean of θ. But the mean is zero except in the $N = 1$ case, which as we saw can be handled directly.] The sum is over nonnegative n_k.

The form of the differential operators is easy to understand if we examine the region near the origin. For class BD, invariant differential operators can be constructed from $\partial/\partial\theta_{ab}$. Invariance under $K = {\rm SO}(N)\times\ {\rm SO}(N)$ implies that near $\theta = 0$, the above operators reduce to

$$\Delta_{2k} = \sum_{\substack{a_1,\ldots,a_k \\ b_1,\ldots,b_k}} \frac{\partial}{\partial\theta_{a_1b_1}} \frac{\partial}{\partial\theta_{a_1b_2}} \frac{\partial}{\partial\theta_{a_2b_2}} \cdots \frac{\partial}{\partial\theta_{a_kb_1}}, \tag{35}$$

$$\Delta'_{p_N} = \sum_{\substack{a_1,\ldots,a_N \\ b_1,\ldots,b_N}} \varepsilon_{a_1\ldots a_N}\varepsilon_{b_1\ldots b_N} \frac{\partial}{\partial\theta_{a_1b_1}} \cdots \frac{\partial}{\partial\theta_{a_Nb_N}}. \tag{36}$$

Here we see that the first $N-1$ operators resemble traces, and were constructed using Kronecker δs, while the last used the Levi-Civita tensor ε. Hence, the first $N - 1$ are invariant under the larger symmetry [the action of any element in ${\rm O}(N)\times\ {\rm O}(N)$], while Δ'_{p_N} is odd under elements of ${\rm O}(N)\times\ {\rm O}(N)$ that have determinant -1. For class DIII, one has, more simply,

$$\Delta_{2k} = \frac{1}{2^{2k}} {\rm tr} \left(\frac{\partial}{\partial\theta} \right)^{2k}, \tag{37}$$

$$\Delta'_{p_N} = \frac{i^N}{2^N} \sum_{a_1,\ldots,a_{2N}} \varepsilon_{a_1\ldots a_{2N}} \frac{\partial}{\partial\theta_{a_1a_2}} \cdots \frac{\partial}{\partial\theta_{a_{2N-1}a_{2N}}}, \tag{38}$$

with similar transformation properties, this time under ${\rm O}(2N)$. We emphasize that in both cases these forms are only valid for small θ, and that for general positions on G/K the expressions become more complicated, in such a way that the operators are invariant under G (although, in local coordinates around any point, they would take the same above form, due to the symmetry of the symmetric space).

The operator Δ_2 is the Laplace-Beltrami operator on G/K in both cases. For the special case $N = 2$, Δ'_{p_N} is also second order. It is a theorem that for an irreducible symmetric space G/K, the Laplace-Beltrami operator is the unique second-order invariant differential operator, up to multiplication by a constant. For $N = 2$, the symmetric spaces for transfer matrices in classes BD and DIII cease to be irreducible. This is basically because of the isomorphisms of Lie algebras so(4)

$\cong$ sl(2)×sl(2), and the symmetric spaces likewise become direct products[21]. There is then a distinct Laplace-Beltrami operator for each factor in this product (each operator involving only derivatives along the directions lying within the factor space). The same essential point was noticed earlier in Ref. 9. [Incidentally, the transfer matrix spaces for all three chiral symmetry classes for $N > 1$ are also reducible, and contain the real line as a direct factor (the position on this line is described by $\sum_i x_i$). Consequently, for all three cases, there are two invariant second-order operators (as pointed out for the orthogonal and unitary cases in Ref. 27), and a drift term along the real line (i.e. a first-order operator) is also invariant under the simply connected group G, but breaks the discrete symmetry which sends $\theta \to -\theta$, or in terms of the radial variables, $x_i \to -x_i$ for all i. In a chain with off-diagonal randomness in the nearest-neighbor hopping, the drift term is produced by staggering the magnitude of the mean hopping[20,28].]

Due to the invariance of the evolution and initial conditions under the subgroup K, only the radial parts of the operators Δ_{2k} and Δ'_{p_N} will be needed in the following. In terms of the radial parts as defined in Appendix A, $\Delta'_{p_N} = N!\Delta_{p_N}$. The angular dependence of $P(\theta; L)$ is trivial, and we can reduce the equation to one for the probability density in only the x_i variables, $P(x; L)$. Because P is a probability density, the relation is $P(x; L) = JP(\theta; L)$. The generalized FP equation for $P(x; L)$ takes the same form as Eq. (34), but with D_{2k} and $N!D_{p_N}$ in place of Δ_{2k} and Δ'_{p_N}, where

$$\begin{aligned} D_{2k} &= J\Delta_{2k}J^{-1}, \\ D_{p_N} &= J\Delta_{p_N}J^{-1}. \end{aligned} \tag{39}$$

We can now describe the fate of the terms of more than second order in the generalized FP equation, as $a \to 0$. For any probability distribution of T_j which has finite second cumulant with coefficient c_2 for θ, we define the mean free path ℓ through Eq. (16), where C_2 depends on the symmetry class, and is chosen so that the coefficient of the second order operator in the FP equation takes the standard form, Eq. (24). We are only interested in $a < \ell$. If θ has nonzero mean, which can occur only for $N = 1$, then we take this mean to be a number times $a/(\gamma\ell)$. The moments $m_{n_1,\ldots,n_N}$ (other than the mean) with total degree $k_{\rm tot} = \sum_{k=1}^{N-1} 2kn_k + Nn_N$ are then of order $(a/\gamma\ell)^{k_{\rm tot}/2}$. If we take the limit $a \to 0$ with $\gamma\ell$ fixed, then all operators of higher order than second drop out, and the usual FP equation is obtained. For $N > 2$, all terms that break the additional symmetry are lost in this limit. For $N = 1$ and 2, a drift term, and an additional second-order term respectively survive in the limit, and these do break the symmetry unless their coefficients are tuned to zero. It has been noticed previously that localization ensues when the drift term is present for $N = 1$, as discussed above[12], and that the non-critical form of the DOS, Eq. (1), results when the additional second-order term is included[9] for $N = 2$. [Similarly, for the three chiral classes, the N transitions, at each of which the behavior is critical, can be reached within the FP approach by tuning the coefficient of the drift term[20].]

It is instructive to realize that the vanishing of all higher-than-second-order terms in the limit $a \to 0$ is connected with the central limit theorem, and also to interpret it in terms of the renormalization group (RG). When the typical values of θ_{ab} for each time step are small (compared with $1/\sqrt{N}$), then in the product of T_j over any time interval that is not too long (less than about ℓ — not $\gamma\ell$, as we explain below), to a good approximation the θs simply add (as matrices). G/K is a smooth manifold, and has a finite radius of curvature at any point. Over this time interval, the random position on G/K has been displaced by an amount less than the radius of curvature. Hence we are essentially looking at a random walk in a flat space. The central limit theorem for sums of independent, identically-distributed random variables then tells us that the sum becomes Gaussian distributed in the limit of many variables, provided the mean and variance for each variable are finite. This means that the probability density for the sum effectively obeys a FP equation with no higher order terms over a long enough time interval. In this flat space case, this can be understood in terms of the generalized FP equation as saying that as the time gets longer, the distribution spreads, and the values of the higher derivatives become smaller relative to the second derivative, and so can be neglected. In renormalization group language, the terms of higher order than second are irrelevant (in this flat space regime), as they become less important at larger time scales. However, if we reverse this process and go to shorter time scales, the higher derivatives get larger. If we take the full generalized FP equation literally and use it for the evolution of an initial δ function for arbitrarily short times, then it ceases to make sense. Of course, the derivation from discrete time steps implies that the equation for time step a certainly cannot be used for times less than a. It effectively describes the process "at the time scale a". The fact that the irrelevant operators have coefficients that contain positive powers of a (the short-time or ultraviolet cutoff) is typical for irrelevant operators. As always in the RG, "relevant" and "irrelevant" refer to linearized scaling behavior near a particular fixed point. In the present case, the fixed point is the ballistic fixed-point, which is described by the FP equation, or by using a Gaussian distribution for the increments and taking the $a \to 0$ limit. The $a \to 0$ limit can be taken only if we are prepared to drop the effects of the irrelevant operators.

We now examine the behavior at longer times. As its name implies, the ballistic fixed-point describes very short times where there is no scattering. The mean free path ℓ is the time scale at which scattering, which corresponds to curvature of G/K, first shows up. We are interested in a small, $a \ll \ell$, and all the coefficients of the irrelevant operators can be thought of as small, though we do not need to set them to zero. Over these time

intervals, the central limit theorem applies, and the FP equation gives a reasonable description of the evolution of $P(x;L)$. For $L \ll \ell$, all x_i are small, and $\mathcal{G} \simeq dN$. We should note that the squares of the x_is are the eigenvalues of $\theta\theta^T$, and so have typical values of order N times those of the squares of elements θ_{ab}. As the time L passes through the localization lengths $\xi_N < \xi_{N-1} < \cdots < \xi_2$, typical values of the corresponding x_i pass through the vicinity of 1 and begin to grow linearly (the smallest such length ξ_N is close to ℓ). Their contribution to $\mathcal{G}$ therefore drops, and we are in the Ohmic regime in which the mean and typical $\mathcal{G}$ are of order $dN\ell/L$ ($N\ell/L$ is the number of channels not yet localized at time L). Here there are also fluctuations in $\mathcal{G}$. This crossover behavior is "universal", even at finite N, in the sense that the irrelevant operators can be dropped; the universality arises, as usual in the RG, from the RG flow passing close to a fixed-point, in this case the ballistic fixed point which is reached as $L \to 0$ provided the irrelevant operators are dropped. However, this universality may not be of great interest physically. If one also takes the limit $N \to \infty$ with $\gamma\ell$ fixed, so $\ell \to 0$, before taking $L \to 0$, then one arrives at a "diffusive" fixed point. That is, if N is large but finite, then the RG flow from the ballistic fixed point approaches close to the diffusive fixed point. This universality in the Ohmic regime when the diffusive fixed point is approached as $L \to 0$ is what is usually meant when the term "universal" is used in mesoscopic physics for the behavior of the conductance in a 1D system. The diffusive fixed point will not concern us here, but we point out that the higher-order differential operators are also irrelevant at this fixed point, and so are dropped to obtain this universal behavior also. Then not only the Ohmic regime, but the whole crossover that begins from the diffusive fixed point at $L \to 0$ is universal, and can be studied as in Ref. 4.

When the time L passes approximately the inverse of the last nonzero Lyapunov exponent, that is $L \simeq \xi_2 \simeq \gamma\ell/m_o$, then only x_1 is still not growing linearly. If one uses the FP Eq. (24), then this is where the asymptotic behavior termed critical sets in. Thus the Ohmic regime is a crossover from the diffusive fixed point to a critical fixed point, and ξ_2 is our estimate for the crossover time, $\xi_\times \sim \xi_2 \sim \gamma\ell$. In most other symmetry classes, localization would set in at times larger than ξ_1, so that the localization length ξ is about $\gamma\ell$, in agreement with RG analysis in 1D. In the present case, the analysis of the RG is similar, but the crossover is to a critical rather than a localized fixed point.

We are concerned with the effect of the symmetry-breaking operators in the generalized FP equation on the critical fixed point. We assume that the coefficient of the leading such term (D_{p_N}) is small, but not zero. Then it, and all other terms of orders higher than second, can be dropped when studying the crossovers from ballistic to diffusive to critical. The next question is whether they are important (relevant) at the critical fixed point.

The main points can be understood by retaining in the right-hand side of the generalized FP only the terms containing the differential operators D_2 and D_{p_N}, so that

$$\frac{\partial P(x;L)}{\partial L} = \frac{1}{2\gamma\ell} D_2 P + (-1)^N \frac{c'_N}{a} D_{p_N} P, \tag{40}$$

(note that $m_{0,0,\ldots,1} = (-1)^N c'_N$). Because we are in the regime where all x_i except x_1 are large, and because the asymptotic behavior for $c'_N = 0$ is dominated by x_1, we will attempt to "integrate out" these x_i. In Appendix A, we determine explicitly the asymptotic forms $\widetilde{D}_{2k}$ and $\widetilde{D}_{p_N}$ of the operators D_{2k} and D_{p_N} for classes BD and DIII, in the regime $|x_1| \ll x_2 \ll \cdots \ll x_N$ (but no other assumptions on x_1). Then we define the reduced probability density $P_1(x_1;L)$ for x_1 by integrating x_i, $i \geqslant 2$ over the range $|x_1| \leqslant x_2 \leqslant \cdots \leqslant x_N$. An equation for P_1 can be obtained by integrating the generalized FP equation over the same range. By using the asymptotic form of $P(x;L)$, Eq. (28), (in the derivation of which, the effects of c'_N and other higher-order operators are negligible), we find that in the $\widetilde{D}$'s, the $\partial/\partial x_i$s for $i \geqslant 2$ can be replaced by zero to obtain the corresponding differential operators in x_1 only, which act on P_1. In particular, $\widetilde{D}_{p_N}$ becomes $\propto \partial/\partial x_1$, and still breaks the symmetry $x_1 \to -x_1$. The resulting equation (for either class BD or DIII) is

$$\frac{\partial P_1}{\partial L} = \frac{1}{2\xi_\times} \frac{\partial^2 P_1}{\partial x_1^2} + \kappa \frac{\partial P_1}{\partial x_1}. \tag{41}$$

Here we used

$$\xi_\times = \gamma\ell, \tag{42}$$

$$\kappa = -m_o^{N-1}(N-1)! \frac{c'_N}{a}. \tag{43}$$

Equation (41) has no explicit N-dependence, and in particular is precisely the form obtained for $N = 1$, on dropping operators higher than second order. Also, this justifies the use of the term "critical fixed point" when $\kappa = 0$, since the equation allows a scaling analysis, with $x_1 \sim (L/\xi_\times)^{1/2}$. This fixed point reproduces the critical asymptotic behavior of $\mathcal{G}$ or $\ln \mathcal{G}$, of course.

Hence, the symmetry-breaking terms produced by disorder that is not invariant under the additional symmetry are relevant at the critical fixed point, and there is a phase transition at $\kappa = 0$ between two phases. In the RG, such operators that are irrelevant at a fixed point in the ultraviolet (such as the ballistic or diffusive fixed points), but relevant at a fixed point at a larger time scale (in the infrared) are called *dangerously irrelevant*, because even though they are irrelevant at the ultraviolet fixed point, they cannot be set to zero without some important physics being lost. For $\kappa \neq 0$, Eq. (41) shows that the Lyapunov exponent for x_1 is $|\kappa|$, so that

$$\xi = \frac{1}{|\kappa|} \tag{44}$$

describes the localization length when the typical (i.e. high-probability) behavior of $-\ln \mathcal{G}$ is growing linearly as $2L/\xi$. The behavior at the transition is universal and described by Eq. (41) provided L and $\xi \gg \xi_\times$, *even if N is finite or small so that the crossover from the diffusive to the critical fixed point is not fully universal*. In this regime, the probability density of $y \equiv (-\frac{1}{2}\ln \mathcal{G})/(L/\xi_\times)^{1/2} \geqslant 0$ approaches a universal function of y and $\xi_\times L/\xi^2$, because $\ln \mathcal{G} - \ln(4d)$ is approximately $-2|x_1|$, except near $\mathcal{G} \simeq d$, but that region has negligible probability for large L. The precise result is

$$P(y) = \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\left[y - \left(\frac{\xi_\times L}{\xi^2}\right)^{1/2} \right]^2 \right\} + \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\left[y + \left(\frac{\xi_\times L}{\xi^2}\right)^{1/2} \right]^2 \right\}. \quad (45)$$

The regime of linear growth of the typical value of $-\ln \mathcal{G}$ sets in at the time scale $L \sim \xi^2/\xi_\times$, which is much larger than ξ under our assumptions that define the scaling regime. At smaller time scales, it grows only as $C\sqrt{L/\xi_\times}$, for some constant C, as described for the critical $\kappa = 0$ case using the N channel FP equation.

All the moments of $\mathcal{G}$ are of order $e^{-(\xi_\times L)/(2\xi^2)}\sqrt{\xi_\times/L}$, throughout the critical regime. (This behavior differs from the "log-normal" behavior in a one-channel wire in most other symmetry classes, because usually the analogs of $\xi_\times$ and ξ are proportional.) Thus, while $\xi_{\rm typ} = \xi = 1/|\kappa|$ describes the rate of linear decay of the typical log conductance,

$$\xi_{\rm mean} = 4\xi^2/\xi_\times \quad (46)$$

describes the exponential decay of the mean conductance. Similar behavior $-L/\xi_{\rm typ}$ and $e^{-L/\xi_{\rm mean}}$ can be inferred for the typical behavior of the logarithm, and for the mean behavior, respectively, of the eigenfunctions at zero energy, as these determine the transmission through the system. This behavior also applies to mean values of the Green's functions, as we will see in the following section. The mean localization length is much longer then the typical one, because the average transmission is dominated by the rare events that produce $|x_1| < 1$.

We can check the effect of all other terms in the generalized FP equation in the same way. We find that in asymptopia, all $\widetilde{D}_{2k}$ reduce to operators $\partial^{2k}/\partial x_1^{2k}$. At the critical fixed point, operators of order higher than two are again irrelevant, and the only other term that should be kept is a correction of second order coming from $(\widetilde{D}_{p_N})^2$, which gives a small correction to $1/\xi_\times$. As an aside we mention that a similar analysis could be done in the (Ohmic) crossover regime, by integrating out only the subset of x_i that typically are growing exponentially (this means introducing a somewhat arbitrary cutoff on i, since there is not a clear separation of large and small x_i in this case, unlike in the vicinity of the critical fixed point). This thinning out of degrees of freedom, a typical method in the RG, would show explicitly how the RG acts in this regime. The variables x_i that are growing linearly correspond to channels in which transmission is decaying exponentially, like massive degrees of freedom in a field theory.

We can combine the forms obtained here for the mean and typical localization lengths with the values of the moments that would be obtained for the non-Gaussian probability distribution of each θ_j at the lattice scale a. If the distribution has a fixed form that scales with the width $\sim (a/\gamma\ell)^{1/2}$, with a parameter h for the strength of the leading symmetry-breaking cumulant in these units (h stays fixed as $a \to 0$), so that $c'_N = h(a/\gamma\ell)^{N/2}$ for $N \geqslant 2$, then we find that

$$\frac{\xi_{\rm typ}}{\xi_\times} = \frac{1}{m_o^{N-1}(N-1)!|h|}\left(\frac{a}{\gamma\ell}\right)^{-(N/2-1)}, \quad (47)$$

while $\xi_{\rm mean}/\xi_\times = 4(\xi_{\rm typ}/\xi_\times)^2$. This illustrates how the (dangerous) irrelevance of the symmetry-breaking perturbation at the ballistic and diffusive fixed points may translate into very large localization lengths, especially as $a/\gamma\ell \to 0$ with N large but fixed, and $\xi_\times$ and h fixed. Note that the limit $N \to \infty$ with $\xi_\times$ fixed is appropriate for studying the diffusive regime and the crossover; the localization lengths appear to vanish in this limit if h and $a/\gamma\ell$ are held fixed, however the analysis assumed that $\xi_{\rm typ} \gg \xi_\times$ (i.e. small $|h|$). That is, if $|h|$ is sufficiently large, localization will occur without the RG flow approaching the vicinity of the critical fixed point analyzed above.

IV. SUPERSYMMETRY SOLUTION

In this section, we discuss the application of a supersymmetry method to the problem of localization in a superconducting wire in class BD. The details of the calculations are provided in Appendix C, and here we only give a summary of the results. The technique is effectively a finite-N analog of the 1D supersymmetric nonlinear σ model, which arises in the $N \to \infty$ limit. The forward and backward directions along the wire are referred to here as "up" and "down", and after averaging over disorder in the subgroup K, the states are in a tensor product space of two "superspins", which are the representation spaces of the superalgebra osp(2|2), R and $\bar{R}$ for up and down, respectively. After averaging, these are coupled by the backward scattering (θ) part of the transfer matrix M. The usual Gaussian disorder, invariant under the additional symmetry, produces a coupling of the superspins that corresponds to the usual Heisenberg coupling familiar from magnetism. The leading symmetry-breaking cumulant produces an additional term with the coefficient κ, given by Eq. (43). When it is nonzero, it produces localization. A dimensionless parameter $|\lambda| = \xi_\times/(2\xi_{\rm typ})$ that enters the calculations in Appendix C is proportional to $|\kappa|$.

The properties of the effective superspin problem are easily translated to the properties of the disordered wire. In particular, the mean localization length $\xi_{\rm mean}$ in the wire is inversely proportional to the energy gap above the ground state at zero "energy" (conjugate to "time", i.e. distance along the wire) in the superspin problem. The DOS in the wire is proportional to the expectation value (in the ground state) of the "staggered magnetization" for the two superspins, as a function of true energy. We have solved the superspin problem and have found its eigenstates and the spectrum. (We present results for N even only. The case $N = 1$ was done previously[29], using results from Ref. 28.) The excited states are separated from the ground state by a gap that turns out to be

$$E_{\rm gap} = 2/\xi_{\rm mean}, \tag{48}$$

where $\xi_{\rm mean}$ is again given by Eqs. (46, 42–44) for $m_o = 1$. This implies that all moments of $\mathcal{G}$ will again decay as $\sim e^{-2L/\xi_{\rm mean}}$ (when $E_{\rm gap}$ is nonzero), confirming the result of the previous section by a very different method.

We also determine the low-energy DOS $\nu(\epsilon)$ in the wire both at and near the critical point. At criticality ($\lambda = 0$), we obtain Dyson's singularity (2). Close to the critical point ($|\lambda| \ll 1$) we get the divergent DOS as in Eq. (1) with exponent $\delta = 1 - 4|\lambda|$. At criticality but at finite ϵ, the behavior can be characterized in terms of a typical localization length $\xi_{\rm typ} \sim |\ln \epsilon|$ as $\epsilon \to 0^+$, and a mean localization length $\xi_{\rm mean} \sim \ln^2 |\epsilon|$. This distinction has been emphasized by D. Fisher in his study of random spin chains[30]. The same forms that we find here were obtained using supersymmetry techniques in Ref. 28 for a one-channel model in the chiral orthogonal class (BDI in AZ classification). We comment further on the relation of the transitions in 1D in different symmetry classes in Section VI below. It is interesting that two analogous length scales emerged rather easily from the transfer matrix analysis (especially if one considers only $N = 1$). For the random hopping problem (class BDI with one channel), an approach using the transfer matrix and a scaling argument[31] reproduced Dyson's DOS at criticality. It appears that one can think of the DOS integrated from 0 to ϵ (which can be obtained from the transfer matrix[32]) as being of order $1/\xi_{\rm mean}$ by scaling, as it is really a mean Green's function property; then differentiating reproduces Dyson's result.

It was emphasized by Bocquet *et al.*[11] that the target space of the nonlinear σ model for classes BD and DIII is disconnected (it is OSp(2|2)/U(1|1) for the simplest case in class BD, and the bosonic sector is O(2)/U(1), which has two components). Thus there can be domain walls at which the field jumps from one component to the other. In terms of superspins, the representation R splits into subspaces called R_0 and R_N in Appendix C, which are irreducible representations of the superalgebra osp(2|2) (there is a similar decomposition of $\bar{R}$). Under the action of the superalgebra, the orbits of the highest or lowest weights in these subspaces form supermanifolds which are the two connected components of OSp(2|2)/U(1|1), so that states in these subspaces correspond to being on one or other component of the supermanifold. The interesting point is that the term that breaks the additional symmetry produces a term in the superspin Hamiltonian that flips between these two subspaces, and thus acts as a domain wall as a function of time (the domain wall is simply a point in 1D). Hence we can conclude that domain walls are essential to produce localization. This has been proposed before[11,12], but now we have explicit results for arbitrary numbers of channels in 1D, and in our framework we know the precise form of the effect the domain wall produces (though we have not analyzed the $N \to \infty$ limit).

V. RANDOM FLUXES

So far, our analysis has been performed in the continuous model introduced in Section II, in which the transfer matrices T_j that are multiplied together for each time step are in the connected group G, and the disorder is weak so each T_j is close to the identity. But the symmetry conditions that define classes BD and DIII do not impose this continuity. They allow the transfer matrix, or its increments T_j, to belong to a larger group $\widehat{G}$ in each case, with $\widehat{G} = {\rm O}(N, N)$ for class BD, $\widehat{G} = {\rm O}(2N, \mathbb{C})$ for class DIII. These groups include matrices of determinant ± 1, and so are not connected. Such transfer matrices may then be expected to arise when the model is not continuous, for example lattice (or tight-binding) models. The negative determinant may be interpreted as saying that when a particle propagates around a closed loop (going first forwards along the wire, then backwards by a different path), it may encircle a magnetic flux of half a quantum, often referred to as a flux of π. Thus we term this the inclusion of random fluxes (of π) in the model (it would perhaps be more accurate to call these random vector potentials). In this section, we analyze the effects of these. The analysis is straightforward in view of the previous results.

It is convenient to consider the following simple model, that is based on the continuous one used up to now. We use a small density (along the wire) of random fluxes. That is, generically the evolution is by T_js in G with weak disorder, distributed as before. But at random times, we also insert a matrix of negative determinant. Such a matrix could be an arbitrary element of $\widehat{G}$. One may wish to view it as a T_j close to the identity, times some element with a negative determinant. As we factor out the T_j close to the identity anyway, we may as well choose the extra matrix to belong to a maximal unitary subgroup $\widehat{K}$ of $\widehat{G}$, which contains K. Relative to the same basis we have used throughout, which refers to the decomposition of M' into in and out subspaces, the natural choice for $\widehat{K}$ is ${\rm O}(N)\times{\rm O}(N)$ or ${\rm O}(2N)$ for classes BD and DIII, respectively (other choices for K or $\widehat{K}$ are isomorphic to these). With this choice, our remarks above about picking up a

phase -1 on propagating around some loops describe exactly what takes place, up to the effect of multiplication by some element of K (if we neglect the disorder outside K which is small over a small time-step).

We note that for class BD, $\widehat{K}$ has four connected components, as may be seen by writing its elements in the form of Eq. (17) but with k_1 and k_2 in $\mathrm{O}(N)$. Then the four choices of signs for the determinants of k_1 and k_2 label the four components. For class DIII, there are just two connected components of $\widehat{K}$. Because $\mathrm{O}(N)$ is a semidirect, not a direct, product of $\mathrm{SO}(N)$ and $\mathbb{Z}_2$ when N is even, there is no unique (i.e. invariant under K) choice of negative determinant matrix in $\widehat{K}$ for class DIII or for class BD for N even. For class BD with N odd, there are two possible natural choices. If we write elements of $\widehat{K}$ in the form of Eq. (17) but with k_1 and k_2 in $\mathrm{O}(N)$, then we can take one of k_1, k_2 equal to I_N, the other equal to $-I_N$.

As time evolves (i.e. L increases), the transfer matrix M evolves on $\widehat{G}$ by diffusion as before, but now also with occasional jumps between the different connected components of $\widehat{G}$, due to the occasional insertion of a negative determinant element of $\widehat{K}$ into the product of transfer matrices for small steps. Modulo multiplication on the right by elements of K, M evolves on the coset space $\widehat{G}/K$, which has four or two connected components for classes BD and DIII respectively, usually by diffusion but with occasional jumps to another component. Modulo the left action of K on these symmetric spaces, evolution is on four (two) copies of the double coset space $K\backslash G/K$ for class BD (DIII, respectively). For class BD, the distinction between components with the same sign for the determinant appears to be unimportant, so we disregard it. The important point is that there is a correspondence between points on the two copies of $K\backslash G/K$, and when a jump occurs, it is between two corresponding points. If we parametrize the two connected spaces under this correspondence by the usual coordinates x_i, then the only difference between evolution on the two components is that in the generalized FP, which applies between jumps, the terms that break the symmetry $x_1 \to -x_1$ have opposite signs on the two components.

We may now consider the effect of the random fluxes on the wire over very long time (length) intervals that contain many jumps. If the jumps are correlated in pairs, so that the system spends most of its time on one component (say, the one in which M is in the identity component), then there will be a net effect of the symmetry breaking terms. In the critical regime, there will be a net drift term. In this case, a transition as we have described it in the previous sections will still occur, and we expect the same universal properties. This is what occurs in the lattice models considered in Refs. 7,17.

On the other hand, if the jumps are uncorrelated, or more generally if equal time is spent on both components, then the average drift will be zero. In this case, localization is suppressed and the system remains critical. (We note that both types of behavior were observed in 1D $N = 1$ models in Ref. 12.) In this case, the results can also be obtained by another approach. After moderate time intervals, the transfer matrices T_j become uniformly distributed over the *disconnected* angular group $\widehat{K}$. We can treat this situation as we did for the connected group K in the continuous model, which leads us to consider the coset spaces $\widehat{G}/\widehat{K}$ and the double cosets $\widehat{K}\backslash\widehat{G}/\widehat{K}$. These are connected spaces, and $\widehat{G}/\widehat{K}$ is the same as G/K. The Weyl group inside $\widehat{K}$ now acts on the radial coordinates x_i by permutations and arbitrary sign changes, so they lie in the Weyl chamber $0 < x_1 < x_2 < \cdots < x_N$, which is the quotient of the previous C by the operation $x_1 \to -x_1$; this parametrizes $\widehat{K}\backslash\widehat{G}/\widehat{K}$. In this case, the generalized FP equation on $\widehat{G}/\widehat{K}$ can contain only differential operators invariant under $\widehat{K}$, and so Δ_{p_N} is not allowed, and its place among the generators of the algebra of invariant differential operators is taken by Δ_{2N} which is of order $2N$. It is now clear from our preceding analysis that in this case there can be no relevant perturbation of the critical fixed point, and the system remains critical, with behavior as described in Ref. 4.

We can also see how these effects would emerge in the supersymmetry framework for class BD (see Appendix C for terminology). If the angular variables must be averaged over $\widehat{K}$, then this produces a constraint that the states on the up (down) sites must be in R_0 ($\bar{R}_0$). Hence there can be no λH_1 term in the Hamiltonian, and in this case all states will be extended. Thus, suppressing domain walls suppresses localization. (Again, this was discussed previously for 2D, and for $N = 1$ in 1D[11,12].)

For the chiral symmetry classes, only the chiral orthogonal class, BDI, possesses a disconnected transfer matrix group $\widehat{G} = \mathrm{GL}(N, \mathbb{R})$, while $\widehat{K} = \mathrm{O}(N)$, and both of these groups have two connected components. However, unlike the cases of classes BD and DIII, in this case $\widehat{K}$ acting by conjugation on the T_j does not reverse the sign of all x_i (the desired additional symmetry operation). Thus the fluxes have apparently a trivial effect, and the disconnectedness of $\mathrm{GL}(N, \mathbb{R})$ is like the factor of 2 in the four connected components of $\mathrm{O}(N, N)$ that likewise had no apparent physical consequences.

VI. SUPER-UNIVERSALITY

Our results have established universality near the critical point, at least for the conductance and the mean density states, in the sense that the universal properties do not depend on the number of channels, or the details of the disorder, within the symmetry classes BD and DIII. Similar results were already known for the orthogonal, unitary, and symplectic chiral symmetry classes (classes BDI, AIII, and CII, respectively). More surprising is that the effective FP equation for the transfer matrix near the critical point, Eq. (41), is the same for all five classes, which implies that the scaling forms for the distributions

both of the conductance and of its logarithm are the same for all five classes. Moreover, the mean density of states also has the same form, first found by Dyson[8] for one case, in all five cases. This suggests that there is "super-universality", meaning that *all five transitions are in the same universality class*, even though they lie in distinct symmetry classes. That is, *all* corresponding universal properties should be the same in the critical region in each class. It is by now clear that in each symmetry class the universal properties of the transition can be obtained from the $N = 1$ case. It is of interest to prove the super-universality directly, and not only for properties related to the transfer matrix. Here we point out that this can be done within lattice models for $N = 1$; these do *not* include random fluxes.

Some results in this direction exist; the equivalence of an $N = 1$ model in class BD, which possesses a transition, with one in class BDI was pointed out in Ref. 29. This can be formulated as follows within a 1D tight-binding chain: a Hamiltonian that consists solely of random imaginary nearest-neighbor hopping (which lies in class BD with $N = 1$) is equivalent by a simple (disorder-independent) gauge transformation to a model with real random hopping (which lies in class BDI with $N = 1$).

Now, for each of the three chiral classes, the one-channel Hamiltonian can be taken to consist solely of nearest-neighbor hopping that, when hopping to the right on any link, is a random positive real number (which we call the magnitude of the hopping) times a random element of $\mathbb{Z}_2$ (i.e. ± 1), U(1) (i.e. a complex number of modulus 1), or SU(2) (i.e. a 2×2 unitary matrix of determinant 1—in this case only, we consider particles with spin 1/2), for the orthogonal, unitary, and symplectic cases, respectively. Clearly, we may perform a randomness-dependent gauge transformation that eliminates the group-valued factors and makes the hopping real and positive everywhere (we assume open, not periodic, boundary conditions). Then the spectrum of the Hamiltonian (for fixed disorder) is the same in each class, and independent of the values of the group-valued disorder. The Green's functions are related by the gauge transformations. If we assume the magnitude of the hopping is statistically independent of the group-valued factor on each link (and the random variables on different links are all independent), then gauge-invariant moments of Green's functions are equal for the different classes. These statements are independent of the distributions for the disorder, apart from the independence properties already stated. The disorder could be weak, consisting of small random corrections to a constant (or more generally, staggered) real positive hopping. This model is useful in taking the continuum limit to establish contact with other models in the literature. The disorder could also be taken with the group-valued factors uniformly distributed over the relevant group, on each link. In this case, the "angular" (group-valued) disorder is strong, though the disorder in the magnitude could still be weak. In all cases, it follows that all universal properties at the transitions in these classes are the same. (It would be surprising if these results were not already known for the chiral classes.)

This leaves only a mapping for a 1D, $N = 1$ model in symmetry class DIII into either class BD or one of the chiral classes to be found. It turns out that there is a mapping to the chiral unitary (AIII) class that parallels that of class BD to the chiral orthogonal class. We take a tight-binding chain with two basis states per site. Class DIII Hamiltonians can be considered as lying in class BD, but with additional restrictions. Thus we may begin with a purely imaginary (and Hermitian) Hamiltonian on this chain, and as usual we expect that nearest-neighbor hopping will be sufficient for our purposes. On examining Altland and Zirnbauer[2], or Motrunich *et al.*[7] [especially their equation (5)], we see (further explanation is given below) that a Hamiltonian in class DIII can be obtained if there are no on-site terms, and if the hopping matrix between neighbors (going to the right) takes the form i times a positive real number, times an orthogonal matrix in O(2) with negative determinant. By a disorder-independent gauge transformation, this can be brought to a real positive number times an SO(2) matrix. Since this is equivalent to a complex number, we see that the Hamiltonian is equivalent to two copies (one the complex conjugate of the other) of a Hamiltonian in the chiral unitary class. Hence our proof is complete, and all the universal properties of the transitions in these five symmetry classes are the same.

We append here a more detailed explanation of the structure of the $N = 1$ class DIII hopping Hamiltonian. Let the components of the state vector be y_i, with y_{2j-1}, y_{2j} belonging to the same site j, $j = 1, \ldots, M$ (in this section, M is a positive integer). Then the Hamiltonian must be an antisymmetric imaginary $2M \times 2M$ matrix, i.e. an element of the Lie algebra so($2M$) of SO($2M$). By definition[2], a Hamiltonian in class DIII lies in so($2M$)/u(M). To understand the action of u(M), we form the combinations $z_j = y_{2j-1} + iy_{2j}$. If all y_i are real (an assumption which does no harm), then the similar combinations $\overline{z}_j = y_{2j-1} - iy_{2j}$ are simply the complex conjugates of z_j. Now a Hamiltonian that is an $M \times M$ Hermitian matrix acting on the vector of z_j can be viewed as an antisymmetric matrix acting on the y_i. These parametrize a linear subspace of so($2M$), and matrix elements of this form are supposed *not* to be present. That means there are no onsite terms, and that any non-zero nearest-neighbor terms must violate the complex structure, effectively mapping z_j to a complex number times $\overline{z}_{j+1}$. In the y_i components, this yields the positive real magnitude times i times a 2×2 orthogonal matrix with determinant -1, as claimed above. The nearest-neighbor structure is essential to the equivalence, as the set of all Hamiltonians in class DIII is certainly not the same as that for AIII, according to the symmetry classification. However, there are also Hamiltonians with longer-range hopping that lie in both classes, and they involve the bipartite lattice structure.

We emphasize that the Hamiltonians here for $N = 1$ channel in classes BD and DIII do not describe superconductors in any simple direct way, as one finds that N is even in those models. But they are significant as we have shown earlier that $N = 1$ controls the critical properties in all cases.

The equivalence of the universality classes for the transitions is not spoiled by the fact that for classes BD and DIII only, one can include random fluxes that prevent the system from becoming localized (at $\epsilon = 0$). This only affects whether or not the relevant perturbation that takes the system off the critical point is possible. The properties at criticality (such as the DOS) are still the same.

The equivalence of the $N = 1$ models considered here suggests that they may possess additional symmetries, especially when viewed within the supersymmetry formalism. The corresponding nonlinear sigma models are certainly not the same, but these correspond to the $N \to \infty$ limit. Apparently, their $N = 1$ analogs must be equivalent. One example of this phenomenon is known[29].

VII. CONCLUSION

We have shown that zero-energy quasiparticle states in a long disordered superconducting wire with broken spin rotation invariance and an arbitrary number (N) of channels are generically localized, both in cases with and in cases without time-reversal invariance. This leads to exponential decay of the thermal conductance of the wire with its length. Within a model with disorder invariant under the "angular" symmetry group K, there are two localized phases, separated by a critical point which can be reached by tuning a parameter characterizing the disorder distribution in the wire. One of our central results is a universal scaling form for the probability distribution of the logarithm of the conductance near criticality, Eq. (45), which exhibits two different length scales, which correspond to the localization lengths for the mean conductance and for the typical log conductance. We also found the mean density of states at low energies both at and near criticality, which agree with Eqs. (1), (2). Our results are in full agreement with those of Motrunich *et al.*[7], who considered the density of states in models with $N = 2$ or 4 channels. Our results confirm that the universal results are independent of the number of channels. We showed that the universality classes for the transitions in the two symmetry classes are the same, and the same as those in the chiral symmetry classes. We also showed how the presence of random vector potentials or fluxes may suppress localization, leaving the system critical. Our analysis explains why the off-critical behavior was not detected in some earlier work that used the FP eq.[4,9], as the use of a continuum ($a \to 0$) limit, or of K-invariant Gaussian disorder, leads to the loss of the relevant perturbation if $N > 2$.

One issue we have left unresolved is the nature of the possible multicritical point found within some class BD models[7]. Another more technical issue is that our analysis used the generalized FP equation, which was derived on the assumption that all moments of the increments are finite. It would be desirable to weaken this assumption as far as possible (as in the proof of the usual central limit theorem, which requires only that the second moment of the increments be finite).

We want to emphasize here that our results for the critical regime are universal, regardless of the number of channels, and thus regardless of whether the diffusive fixed point is reached as $L \to 0$, which occurs only if $N \to \infty$. The critical (scaling) regime is defined by the condition that the typical localization length ξ be much larger than the crossover length $\xi_\times$ from the short length behavior; the critical point corresponds to $\xi = \infty$. $\xi_\times$ may be of order the mean free path ℓ, or much larger $\sim N\ell$ in the many-channel case in which short-length behavior is diffusive (Ohmic). We used a model description with disorder invariant under the angular subgroup K. More general models do not have this property at short scales, but it is expected that the angular variables in the transfer matrix rapidly become uniformly distributed, so that our results should be universal for all such models, in the scaling regime stated.

We also wish to emphasize that our results are generic for classes BD and DIII. As in the three chiral symmetry classes (BDI, AIII, and CII), states are generally localized, but there can be phase transitions as a parameter is tuned. This critical behavior is intrinsic to these symmetry classes; the behavior must be described by a parameter that specifies the distance from the critical point, as well as a mean free path. This may seem to contradict some widely-held assumptions for 1D disordered wires, that the symmetry class determines everything, and that the presence of any other parameter means "non-universality". However, in the general theory of critical phenomena, universal scaling functions of more than one variable are common. The necessity of using other parameters in addition to the conductance at the scale of the mean free path to describe universal phenomena should be familiar from the example of the quantum Hall effect in two dimensions, where the additional parameter is the Hall conductance. Symmetry classes are not universality classes; universality occurs when the RG flow passes near a fixed point, and there may be more than just the ballistic, diffusive, and localized fixed points within a given symmetry class. Which universality class occurs in a given model within a symmetry class depends on other aspects of the structure of the model. Examples of this have been previously observed in class BD in two dimensions[12,29].

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APPENDIX A: INVARIANT DIFFERENTIAL OPERATORS ON G/K

In this Appendix we gather necessary information on the spaces of transfer matrices for classes BD and DIII, viewed as symmetric spaces G/K. A standard detailed reference for analysis on symmetric spaces and invariant differential operators is Ref. 22. The appendices in Ref. 23 contain an accessible introduction to root systems and symmetric spaces.

1. Class BD

In this case the space of transfer matrices, of type DI in Cartan's classification, is $M = G/K$, where

$$G = \mathrm{SO}_0(N,N), \qquad K = \mathrm{SO}(N) \times \mathrm{SO}(N). \tag{A1}$$

The Lie algebra of the group G is $\mathrm{Lie}(G) = \mathrm{so}(N,N)$, and it admits the Cartan decomposition:

$$\mathrm{so}(N,N) = \mathrm{so}(N) + \mathrm{so}(N) + \mathcal{P}, \tag{A2}$$

where the subspace $\mathcal{P}$ consists of matrices of the form

$$\begin{pmatrix} 0 & \theta \\ \theta^T & 0 \end{pmatrix}, \tag{A3}$$

with a real $N \times N$ matrix θ.

Within the space $\mathcal{P}$ there is a maximal Abelian subspace $\mathcal{A}$, and one convenient choice is the set of matrices of the form

$$a_X = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}, \quad X = \mathrm{diag}(x_1, \dots, x_N), \quad x_i \in \mathbb{R}. \tag{A4}$$

Let us choose a canonical orthonormal basis $\{e_1 \dots e_N\}$ in the space $\mathcal{A}^*$ dual to $\mathcal{A}$ (this space is isomorphic to $\mathbb{R}^N$ as a vector space) so that

$$e_i(a_X) = x_i. \tag{A5}$$

Then the set of roots α of the pair $(\mathrm{Lie}(G), \mathcal{A})$ is

$$\Sigma = \big\{ \pm e_i \pm e_j, \quad 1 \leqslant i < j \leqslant N \big\}. \tag{A6}$$

These vectors form a root system of type D_N. All the roots have the same length, and their multiplicities m_α are all equal to

$$m_o = 1. \tag{A7}$$

It is convenient for our purposes to choose as positive the roots

$$\Sigma^+ = \big\{ e_j \pm e_i, \quad 1 \leqslant i < j \leqslant N \big\}. \tag{A8}$$

This somewhat non-standard choice leads to the following dominant Weyl chamber (note that x_1 may be positive or negative):

$$|x_1| < x_2 < \ldots < x_N. \tag{A9}$$

This makes the middle element in the Cartan decomposition (7) unique.

The half-sum of the positive roots (A8) is

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha = \sum_{i=1}^N (i-1) e_i,$$
$$\rho(x) \equiv \rho(a_X) = \sum_{i=1}^N (i-1) x_i. \tag{A10}$$

The density function (usually denoted $\delta(x)$ in the mathematical literature[22])

$$J(x) = \prod_{\alpha \in \Sigma^+} \big(\sinh \alpha(a_X) \big)^{m_\alpha}$$
$$= \prod_{i<j} \sinh(x_j + x_i) \sinh(x_j - x_i) \tag{A11}$$

is exactly the Jacobian which appears in the Fokker-Planck equation (24).

The Weyl group W of the root system of type D_N is the semidirect product of the symmetric group S_N (permuting the e_i) and $\mathbb{Z}_2^{N-1}$ (acting by an even number of sign changes $e_i \to -e_i$). The ring of polynomial invariants of W is generated by

$$s_2, s_4, \dots, s_{2N-2}, p_N, \tag{A12}$$

where

$$s_{2k}(x) = \sum_{i=1}^N x_i^{2k}, \qquad p_N(x) = \prod_{i=1}^N x_i. \tag{A13}$$

We can also form W-invariant differential operators on $\mathcal{A}$ replacing x_i by $\partial_i \equiv \partial/\partial x_i$ in these expressions. The operators obtained in this way,

$$s_2(\partial), \dots, s_{2N-2}(\partial), p_N(\partial) \tag{A14}$$

are local analogs (valid near $x_i = 0$ for all i) of the generators

$$\Delta_{s_2}, \dots, \Delta_{s_{2N-2}}, \Delta_{p_N} \tag{A15}$$

of the algebra of K-radial parts of the globally G-invariant differential operators on G/K. This may be understood directly, by comparing these expressions with Eqs. (35, 36), transformed to radial variables, and dropping the angular parts.

In general, only $\Delta_{s_2} \equiv \Delta_2$, the radial part of the Laplace-Beltrami operator is known explicitly:

$$\Delta_2 = \sum_{i=1}^N J^{-1} \frac{\partial}{\partial x_i} J \frac{\partial}{\partial x_i}. \tag{A16}$$

Conjugation of this expression by the density function leads to the operator D_2 appearing in the FP equation, and the other operators $D_{s_{2k}} \equiv D_{2k}$ and D_{p_N} are obtained similarly; see Eqs. (27) or (39).

For the purposes of this paper we do not need explicit expressions for the Ds, but only their asymptotic limits in the regime of a long wire, when the radial coordinates are widely separated, $|x_1| \ll x_2 \ll \ldots \ll x_N$. In this regime many simplifications occur. First, the density function can be approximated as

$$\begin{aligned} 2^{\sum m_\alpha} J(x) &= e^{\sum m_\alpha \alpha(a_X)} \big(1 + O(e^{-2\alpha_{\min}(a_X)})\big) \\ &\approx e^{2\rho(x)} = \prod_i e^{2(i-1)x_i}. \end{aligned} \tag{A17}$$

Secondly, it follows from Theorem II.5.23 in Ref. 22 that the differential operators of Eq. (A15) tend to

$$\begin{aligned} \Delta_{s_k} \to \widetilde{\Delta}_{s_k} &= e^{-\rho(x)} s_k(\partial) e^{\rho(x)} - e^{-\rho(x)} \big(s_k(\partial) e^{\rho(x)}\big) \\ &= \sum_{i=1}^N \Big(\big(e^{-\rho(x)} \partial_i e^{\rho(x)}\big)^k - \big(\partial_i \rho(x)\big)^k \Big), \\ \Delta_{p_N} \to \widetilde{\Delta}_{p_N} &= e^{-\rho(x)} p_N(\partial) e^{\rho(x)} - e^{-\rho(x)} \big(p_N(\partial) e^{\rho(x)}\big) \\ &= \prod_{i=1}^N e^{-\rho(x)} \partial_i e^{\rho(x)} - \prod_{i=1}^N \partial_i \rho(x). \end{aligned} \tag{A18}$$

In the subtracted terms the derivatives act only on $e^{\rho(x)}$, producing x-independent constants so that no non-derivative terms appear in the operators.

Then it follows that the asymptotic forms we need are

$$\begin{aligned} D_{s_k} \to \widetilde{D}_{s_k} &= e^{\rho(x)} s_k(\partial) e^{-\rho(x)} - e^{-\rho(x)} \big(s_k(\partial) e^{\rho(x)}\big) \\ &= \sum_{i=1}^N \Big(\big(e^{\rho(x)} \partial_i e^{-\rho(x)}\big)^k - (i-1)^k \Big), \\ D_{p_N} \to \widetilde{D}_{p_N} &= e^{\rho(x)} p_N(\partial) e^{-\rho(x)} - e^{-\rho(x)} \big(p_N(\partial) e^{\rho(x)}\big) \\ &= \prod_{i=1}^N e^{\rho(x)} \partial_i e^{-\rho(x)} - \prod_{i=1}^N (i-1). \end{aligned} \tag{A19}$$

For class BD we have (see Eq. (A10))

$$e^{\rho(x)} \partial_i e^{-\rho(x)} = \partial_i - i + 1, \tag{A20}$$

so, for example, the FP equation in the asymptotic limit contains

$$\begin{aligned} \widetilde{D}_2 &= \sum_{i=1}^N \big((\partial_i - i + 1)^2 - (i-1)^2 \big) \\ &= \sum_{i=1}^N \big(\partial_i^2 - 2(i-1)\partial_i \big). \end{aligned} \tag{A21}$$

Finally, the operator we are interested in most, D_{p_N}, in the asymptotic limit is replaced by

$$\widetilde{D}_{p_N} = \prod_{i=1}^N (\partial_i - i + 1) = \partial_1 (\partial_2 - 1) \ldots (\partial_N - N + 1). \tag{A22}$$

2. Class DIII

In this case the space of transfer matrices, of type D in Cartan's classification, is $M = G/K$, where

$$G = \mathrm{SO}(2N, \mathbb{C}), \qquad K = \mathrm{SO}(2N). \tag{A23}$$

The Lie algebra $\mathrm{Lie}(G) = \mathrm{so}(2N, \mathbb{C})$ of the group G is usually taken to consist of all antisymmetric complex $2N \times 2N$ matrices. The Cartan decomposition

$$\mathrm{so}(2N, \mathbb{C}) = \mathrm{so}(2N) + \mathcal{P} \tag{A24}$$

coincides with the decomposition of antisymmetric complex matrices into real (for $\mathrm{so}(2N)$) and imaginary (for $\mathcal{P}$) parts.

By the unitary transformation with

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} I_N & i I_N \\ I_N & -i I_N \end{pmatrix} \tag{A25}$$

(I_N is the $N \times N$ unit matrix) the antisymmetric complex matrices may be replaced by matrices of the form

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}, \qquad B^T = -B, \qquad C^T = -C, \tag{A26}$$

where A, B, C are complex $N \times N$ matrices. In this basis the Cartan decomposition can be described as follows. Elements of $\mathrm{so}(2N)$ are matrices (A26) with antisymmetric real part and symmetric imaginary part, and vice versa for the elements of $\mathcal{P}$.

A maximal abelian subspace $\mathcal{A}$ of $\mathcal{P}$ is chosen as the set of matrices

$$a_X = \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix}, \quad X = \mathrm{diag}(x_1, \ldots, x_N), \quad x_i \in \mathbb{R}. \tag{A27}$$

It follows then that the root system in this case is the same as for class BD, Eq. (A6), except that the multiplicities are now

$$m_o = 2. \tag{A28}$$

The positive roots and the dominant Weyl chamber are the same as before, so most of the expressions from the previous subsection apply here as well. Some changes are

$$\rho(x) = \sum_{i=1}^N 2(i-1)x_i, \tag{A29}$$

$$J(x) = \prod_{i<j} \sinh^2(x_j + x_i) \sinh^2(x_j - x_i). \tag{A30}$$

The main difference, however, comes from the fact (which is true whenever the group G is complex) that when all the roots have multiplicity 2, the function $J^{1/2}(x)$ is a linear combination of exponentials of positive roots. Then

it is possible (see Theorem II.5.37 in Ref. 22) to find explicit expressions for all the radial parts of Eq. (A15):

$$\Delta_{s_k} = J^{-1/2} s_k(\partial) J^{1/2} - J^{-1/2}\big(s_k(\partial) J^{1/2}\big),$$
$$\Delta_{p_N} = J^{-1/2} p_N(\partial) J^{1/2} - J^{-1/2}\big(p_N(\partial) J^{1/2}\big). \quad \text{(A31)}$$

As before, the subtractions remove constant non-derivative terms from the differential operators.

Correspondingly, the generalized Fokker-Planck equation contains

$$D_{s_k} = J^{1/2} s_k(\partial) J^{-1/2} - J^{-1/2}\big(s_k(\partial) J^{1/2}\big),$$
$$D_{p_N} = J^{1/2} p_N(\partial) J^{-1/2} - J^{-1/2}\big(p_N(\partial) J^{1/2}\big). \quad \text{(A32)}$$

As in the previous case of class BD, we do not really need these explicit expressions, and, instead, use the appropriate asymptotic limits:

$$\widetilde{D}_2 = \sum_{i=1}^{N} \big(\partial_i^2 - 4(i-1)\partial_i\big),$$
$$\widetilde{D}_{p_N} = \prod_{i=1}^{N} (\partial_i - 2i + 2)$$
$$= \partial_1(\partial_2 - 2)\dots(\partial_N - 2N + 2). \quad \text{(A33)}$$

APPENDIX B: SOLUTION OF THE FOKKER-PLANCK EQUATION

Equation (24) is supplemented by the initial condition which corresponds to $M = 1$ for $L = 0$. There are also some boundary conditions imposed on the function $P(x; L)$. They follow from the fact that the Fokker-Planck equation (24) has the form of the continuity equation for the probability current S_i (we use the short-hand notation $\partial_L = \partial/\partial L$, $\partial_i = \partial/\partial x_i$):

$$\partial_L P = \frac{1}{2\gamma\ell} \sum_{i=1}^{N} \partial_i S_i, \qquad S_i = \partial_i P - P\partial_i \ln J. \quad \text{(B1)}$$

Then the conservation of the total probability $\int_C dx\, P(x; L)$, where the integration is over the dominant Weyl chamber C, is ensured if the normal components of the current vanish on the boundary of C:

$$S_{\hat{n}}\big|_{x\in\partial C} = 0. \quad \text{(B2)}$$

The boundary ∂C is a union of hyperplanes $x_{i+1} = x_i$ for $i = 1$ through $N - 1$, and also $x_2 = -x_1$. Thus we have, for example,

$$\Big[\big(\partial_i P - P\partial_i \ln J\big) - \big(\partial_{i+1} P - P\partial_{i+1} \ln J\big)\Big]_{x_i = x_{i+1}} = 0. \quad \text{(B3)}$$

In addition, as we will see momentarily, the function $P(x; L)$ itself also vanishes on ∂C, see discussion after Eq. (B6):

$$P(x; L)\big|_{x\in\partial C} = 0. \quad \text{(B4)}$$

The Fokker-Planck equation (24) can be solved in general (for any symmetry class) with the help of the so called spherical transform on the space G/K, see Ref. 33. The solution is expressed in terms of the so called spherical functions $\varphi_k(x)$ labelled by N real parameters $k = \{k_1, \dots, k_N\}$. These functions are the eigenfunctions of Δ_2 with eigenvalue $-k^2 = -\sum_i k_i^2$. The solution for $P(x; L)$ (up to a L-dependent normalization constant $N(L)$) in terms of the spherical functions is an analog of the Fourier transform on the double coset $K\backslash G/K$:

$$P(x; L) = N(L) J(x) \int_{-\infty}^{\infty} \frac{\prod_i dk_i}{|c(k)|^2} \exp\Big(-\frac{Lk^2}{2\gamma\ell}\Big) \varphi_k(x). \quad \text{(B5)}$$

The Harish-Chandra function $c(k)$ appearing in the integration measure is known in all cases, and for the BdG classes is given by

$$c(k) = \prod_{i=1}^{N} \frac{\Gamma\big(i\frac{k_i}{2}\big)}{\Gamma\big(\frac{m_l}{2} + i\frac{k_i}{2}\big)} \prod_{i<j}^{N} \prod_{\pm} \frac{\Gamma\big(i\frac{k_j \pm k_i}{2}\big)}{\Gamma\big(\frac{m_o}{2} + i\frac{k_j \pm k_i}{2}\big)}. \quad \text{(B6)}$$

From the solution (B5) it follows that the distribution density $P(x; L)$ vanishes on the boundaries of the Weyl chamber C, because the Jacobian $J(x)$ does, and because the spherical functions $\varphi_k(x)$ are bounded (see Theorem II.8.1 in Ref. 22).

The spherical functions $\varphi_k(x)$ are known explicitly only for classes A, AIII, CI, and DIII in the AZ classification, and so in these cases there are explicit expressions for $P(x; L)$. For all the symmetry classes, however, the expression (B5) can be significantly simplified in the asymptotic limit of long wires $L \gg \gamma\ell$. In this case the "diffusing" radial coordinates become very large, $x_i \gg 1$, and for such values of x_i the spherical functions tend to (see Theorem II.5.5 in Ref. 22)

$$\varphi_k(x) \approx J^{-1/2}(x) \sum_{s\in W} c(sk) e^{i(sk,x)}, \quad \text{(B7)}$$

where the summation is over the elements s of the Weyl group W, and $(,)$ denotes the scalar product. It is important that for classes BD and DIII this is still valid even if $|x_1|$ is not large. In the limit $L \gg \gamma\ell$, only $k_i \ll 1$ significantly contribute to the integral in Eq. (B5). For small values of k_i the Harish-Chandra function becomes

$$c^{-1}(k) \propto \prod_{i=1}^{N} k_i^{\delta(m_l)} \prod_{i<j}^{N} \big(k_j^2 - k_i^2\big), \quad \text{(B8)}$$

where

$$\delta(m_l) = \begin{cases} 0, & \text{for } m_l = 0, \\ 1, & \text{for } m_l > 0. \end{cases}$$

Then the integral in Eq. (B5) can be done, and with the

use of Eq. (25) one can obtain

$$P(x;L) \approx N(L)\prod_{i=1}^{N}\exp\Big(-\frac{\gamma\ell}{2L}x_i^2\Big)x_i^{\delta(m_l)}(\sinh 2x_i)^{m_l/2} \times\prod_{i<j}^{N}(x_j^2-x_i^2)(\sinh^2 x_j-\sinh^2 x_i)^{m_o/2}. \tag{B9}$$

Further simplification is possible if we use the fact that x_i ($i>1$) are not only large in this regime, but they are also widely separated:

$$|x_1| \ll x_2 \ll \ldots \ll x_N. \tag{B10}$$

In this portion of the dominant Weyl chamber far from the origin and the chamber's boundaries the asymptotic solution (B9) simplifies further to the form given in the main text, Eq. (28).

APPENDIX C: DETAILS OF THE SUSY SOLUTION

The supersymmetry technique is most useful for the calculation of moments of Green's functions. The latter can be viewed as sums over paths, each path contributing a product of factors that are the amplitudes (elements of the Hamiltonian or transfer matrix) for each step of the path. Hence it is natural to represent these sums in terms of Green's functions in a field theory for particles (fermions and bosons) that propagate according to these amplitudes. With equal numbers of fermion and boson components, the partition function is unity, and the averages of products of Green's functions can be calculated[28,34,35,36]. The resulting field theories possess symmetries (supersymmetry) that rotate fermion and boson fields into each other. This symmetry guarantees that the partition function is indeed unity.

In the present case, for class BD, the two directions of propagation along the wire will be viewed as vertical and labeled "up" and "down" respectively. The logarithm of the transfer matrix will effectively become a Hamiltonian that propagates the particles forwards (up). At each time (i.e. position along the wire, as in the main text), the two sets of variables, up and down, will be grouped as two "sites". We will view this system then as a two-site quantum Hamiltonian (in imaginary time). For technical reasons, it is necessary at some stage to treat the cases N even or odd separately. We consider only N even, as it describes superconducting wires, and write $N=2n$ for convenience.

On the up site, we may choose the bosons and fermions b_a, f_a (Latin indices run from 1 to N, and we assume summation over repeated indices) to obey canonical commutation relations. On the down site the bosons $\bar{b}_a$ are canonical, but the fermions $\bar{f}_a$ satisfy the following commutation relation:

$$\{\bar{f}_a, \bar{f}_b^\dagger\} = -\delta_{ab}. \tag{C1}$$

As the dagger † always stands for the adjoint, this relation results in negative square norms for states with odd numbers of $\bar{f}$ (if we choose the vacuum to have positive norm). This is necessary for the cancellations that result in the partition function being unity (before averaging over disorder). (Other choices for where the minus signs go are possible, but do not essentially change any results.) We remark that we could use $\mathcal{N}$ copies of the fermions and bosons ($\mathcal{N}=1,\,2,\,\ldots$), and this is necessary when studying higher moments of Green's function than we will need here.

For the transfer matrices, it is convenient for us here to use the Cartan decomposition for the transfer matrices at the nodes in the form

$$T = kp. \tag{C2}$$

The first factor here $k \in K = \mathrm{SO}(N)\times\mathrm{SO}(N)$ describes the propagation and mixing, or forward scattering, of particle fluxes on the up and down sites (the two factors of $\mathrm{SO}(N)\times\mathrm{SO}(N)$ correspond to the up and down sites respectively). As usual, if we assume the uniform distribution for k with respect to the Haar measure on K, then averaging over this distribution projects the fermionic and bosonic Fock spaces to the subspaces of the singlets of $\mathrm{SO}(N)$ on the up and separately on the down sites.

1. The transfer matrix and the Hamiltonian

The second factor $p \in \exp\mathcal{P}$ in the Cartan decomposition (C2) describes the evolution of the states across a node of the network. The element (14) of $\mathcal{P}$ parameterized by θ gives rise to the transfer matrix

$$p = \exp\begin{pmatrix} 0 & \theta \\ \theta^T & 0 \end{pmatrix} = \begin{pmatrix} \cosh\sqrt{\theta\theta^T} & \theta\dfrac{\sinh\sqrt{\theta^T\theta}}{\sqrt{\theta^T\theta}} \\ \theta^T\dfrac{\sinh\sqrt{\theta\theta^T}}{\sqrt{\theta\theta^T}} & \cosh\sqrt{\theta^T\theta} \end{pmatrix}, \tag{C3}$$

where the matrices $\theta^T\theta$ and $\theta\theta^T$ have non-negative eigenvalues, and the square roots are well defined.

The second quantized version of the transfer matrix p is the evolution operator

$$V = \exp\left(i\theta_{ab}J_{ab}\right), \tag{C4}$$

$$J_{ab} = b_a^\dagger\bar{b}_b^\dagger + \bar{b}_b b_a + f_a^\dagger\bar{f}_b^\dagger + \bar{f}_b f_a. \tag{C5}$$

As before, we assume that the transfer matrices are random with some generic K-invariant distribution $P(\theta)$. As in our analysis of the FP equation, it will be sufficient to consider only the cumulants (16) and (20). Then, averaging V over the distribution of the disorder $P(\theta)$ we write it as

$$[V] = e^{-aH}, \tag{C6}$$

where the effective Hamiltonian H contains products of J_{ab} with cumulants of θ, and a is the time step, as before. The two non-trivial cumulants (16, 20) give two terms in the Hamiltonian (we choose to rescale the Hamiltonian by a factor):

$$H_r = \frac{a}{2c_2} H = H_0 + 2\lambda H_1, \tag{C7}$$

$$H_0 = \frac{1}{4}\delta_{a_1a_2}\delta_{b_1b_2} J_{a_1b_1} J_{a_2b_2} = \frac{1}{4} J_{ab} J_{ab}, \tag{C8}$$

$$H_1 = -\frac{N}{2}\frac{i^N}{(N!)^2}\varepsilon_{a_1\dots a_N}\varepsilon_{b_1\dots b_N} J_{a_1b_1}\dots J_{a_Nb_N}, \tag{C9}$$

where the parameter

$$\lambda = -(N-1)!\frac{c'_N}{2c_2} \tag{C10}$$

is chosen to simplify equations in the following. Note that this dimensionless parameter is related to physical length scales introduced in Section III as $|\lambda| = \xi_\times/(2\xi_{\rm typ})$.

As usual, we should add to the Hamiltonian H_r a regulating term

$$H_\omega = \omega(b_a^\dagger b_a + f_a^\dagger f_a + \bar b_a^\dagger \bar b_a - \bar f_a^\dagger \bar f_a), \tag{C11}$$

which suppresses large boson numbers. ω represents the imaginary part of the energy ϵ in the original random Hamiltonian, and is necessary when studying e.g. the DOS. The full effective (rescaled) Hamiltonian is then

$$H_r = H_0 + 2\lambda H_1 + H_\omega. \tag{C12}$$

2. Supersymmetry algebra osp(2|2), and its representations R and $\bar R$

The subspaces of states that are invariant under SO(N)×SO(N) are infinite-dimensional representations of the superalgebra osp(2|2), which we will see is the symmetry algebra of the problem. We denote by R the representation space on the up site, and by $\bar R$ the one on the down site.

The osp(2|2) symmetry is generated by the bilinears in the fermions and bosons that are invariants of SO(N). The generators of osp(2|2) on the up site (in the representation R) are given by (using notation for the generators from Ref. 37)

$$\begin{aligned}
B &= \frac{1}{2}\left(f_a^\dagger f_a - n\right), & Q_3 &= \frac{1}{2}\left(b_a^\dagger b_a + n\right),\\
Q_+ &= \frac{1}{2}\left(b_a^\dagger\right)^2, & Q_- &= -Q_+^\dagger = -\frac{1}{2} b_a^2,\\
V_+ &= \frac{1}{\sqrt{2}} f_a^\dagger b_a^\dagger, & W_- &= -V_+^\dagger = -\frac{1}{\sqrt{2}} b_a f_a,\\
V_- &= -\frac{1}{\sqrt{2}} f_a^\dagger b_a, & W_+ &= -V_-^\dagger = \frac{1}{\sqrt{2}} b_a^\dagger f_a.
\end{aligned} \tag{C13}$$

Similarly, on the down site (in the representation $\bar R$) the generators are

$$\begin{aligned}
\bar B &= \frac{1}{2}\left(\bar f_a^\dagger \bar f_a + n\right), & \bar Q_3 &= -\frac{1}{2}\left(\bar b_a^\dagger \bar b_a + n\right),\\
\bar Q_+ &= \frac{1}{2}\bar b_a^2, & \bar Q_- &= -\bar Q_+^\dagger = -\frac{1}{2}\left(\bar b_a^\dagger\right)^2,\\
\bar V_+ &= \frac{1}{\sqrt{2}} \bar f_a \bar b_a, & \bar W_- &= -\bar V_+^\dagger = -\frac{1}{\sqrt{2}} \bar b_a^\dagger \bar f_a^\dagger,\\
\bar V_- &= \frac{1}{\sqrt{2}} \bar b_a^\dagger \bar f_a, & \bar W_+ &= -\bar V_-^\dagger = -\frac{1}{\sqrt{2}} \bar f_a^\dagger \bar b_a.
\end{aligned} \tag{C14}$$

The generalization to osp($2\mathcal{N}|2\mathcal{N}$) is straightforward, but not needed here.

Using these generators we now describe the representations R and $\bar R$. The representation R splits into two irreducible representations of the algebra osp(2|2). The first one of these, which we denote as R_0 is a lowest weight irreducible representation with the lowest weight state being the vacuum $|0\rangle$ for all b_a, f_a. Other states are obtained by repeated actions of Q_+ and $V_\pm$. Normalized states obtained in this way are

$$\begin{aligned}
\big|2m, 0\big\rangle &= a_m Q_+^m |0\rangle,\\
\big|2m-1, 1\big\rangle &= b_m V_- \big|2m, 0\big\rangle,
\end{aligned} \tag{C15}$$

where

$$a_m = \Big(\frac{(n-1)!}{m!(m+n-1)!}\Big)^{1/2}, \quad b_m = m^{-1/2}. \tag{C16}$$

The other subrepresentation of R denoted R_N is also a lowest weight irreducible representation of osp(2|2), but now the lowest weight state is

$$\big|0, N\big\rangle = f_1^\dagger f_2^\dagger \dots f_N^\dagger |0\rangle. \tag{C17}$$

Other normalized states in R_N are

$$\begin{aligned}
\big|2m, N\big\rangle &= a_m Q_+^m \big|0, N\big\rangle,\\
\big|2m+1, N-1\big\rangle &= c_m W_+ \big|2m, N\big\rangle,
\end{aligned} \tag{C18}$$

with a_m as in (C16) and

$$c_m = (m+n)^{-1/2}. \tag{C19}$$

The representation $\bar R$ is the dual of R. Again, it is split into $\bar R_0$ and $\bar R_N$ under osp(2|2). Both these subrepresentations are highest weight irreducible representations. In $\bar R_0$ the highest weight state is the vacuum $|\bar 0\rangle$ for $\bar b_a, \bar f_a$, and the other normalized states are

$$\begin{aligned}
\big|\overline{2m, 0}\big\rangle &= a_m \bar Q_-^m |\bar 0\rangle,\\
\big|\overline{2m-1, 1}\big\rangle &= b_m \bar W_+ \big|\overline{2m, 0}\big\rangle,
\end{aligned} \tag{C20}$$

with the same a_m, b_m as in (C16). Note that the states with odd number of fermions $\bar f_a$ have negative squared norms. For example,

$$\big\langle\overline{2m-1, 1}\big|\overline{2m-1, 1}\big\rangle = -1. \tag{C21}$$

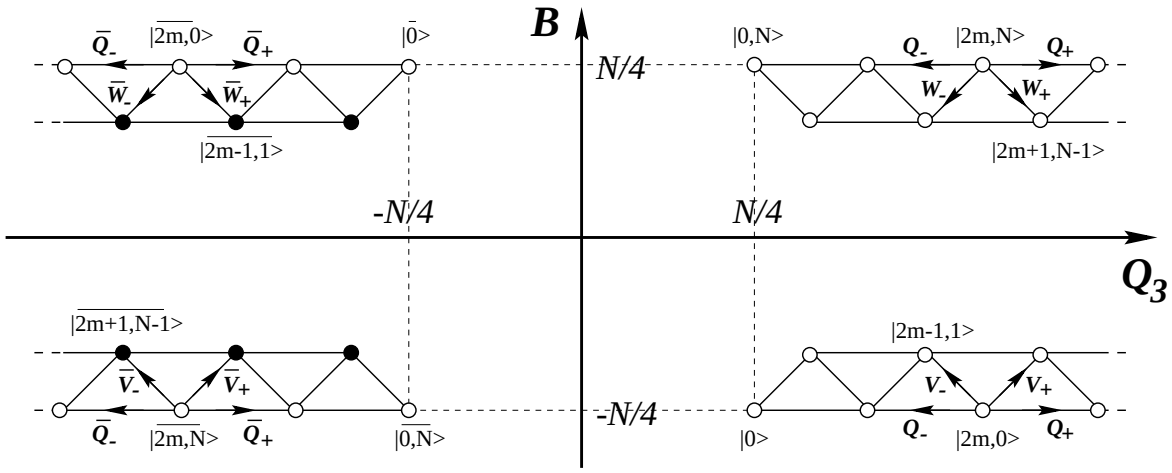

FIG. 2: Weight diagram for states on the representations R and $\bar{R}$.

In $\bar{R}_N$ the highest weight state is

$$\left|\overline{0,N}\right\rangle = \bar{f}_1^\dagger \bar{f}_2^\dagger \dots \bar{f}_N^\dagger |\bar{0}\rangle, \tag{C22}$$

and the rest is

$$\begin{aligned} \left|\overline{2m,N}\right\rangle &= a_m \bar{Q}_-^m \left|\overline{0,N}\right\rangle, \\ \left|\overline{2m+1,N-1}\right\rangle &= c_m \bar{V}_- \left|\overline{2m,N}\right\rangle. \end{aligned} \tag{C23}$$

The states $\left|\overline{2m+1,N-1}\right\rangle$ are negative norm states.

The osp(2|2) weights of the states in R and $\bar{R}$ are shown in Fig. 2. The negative norm states are shown with filled circles. We also show the action of the osp(2|2) generators on the states. We note here that the bosonic part of the supergroup OSp(2|2) under consideration is O(2)×Sp(2, $\mathbb{R}$), and O(2) has two connected components. In the same way that O(2) has two-dimensional irreducible representations [whereas SO(2), and its Lie algebra so(2), have one-dimensional irreducibles], the two subspaces R_0 and R_N together form an irreducible representation of OSp(2|2)[29]. The same is true for the states in $\bar{R}$.

The term J_{ab}^2 in H_0 is an osp(2|2) invariant, and is proportional to the osp(2|2)-invariant product of the generators acting in the representations R and $\bar{R}$. It is convenient to choose the constant $c_2 = 1/2$, and then

$$\begin{aligned} H_0 = {} & 2B\bar{B} - 2Q_3\bar{Q}_3 - Q_+\bar{Q}_- - Q_-\bar{Q}_+ \\ & - V_+\bar{W}_- + W_-\bar{V}_+ + V_-\bar{W}_+ - W_+\bar{V}_-. \end{aligned} \tag{C24}$$

Then it is clear that H_0 acts irreducibly within each of the four subspaces $R_0 \otimes \bar{R}_0$, …, $R_N \otimes \bar{R}_N$.

The term H_1 has a more complicated structure. It is an osp(2|2)-invariant product of N operators on each site. In the representation R we have

$$\begin{aligned} C_+ &= \frac{1}{N!}\varepsilon_{a_1\dots a_N} f_{a_1}^\dagger \cdots f_{a_N}^\dagger, & C_- &= C_+^\dagger \\ D_+ &= \frac{1}{N!}\varepsilon_{a_1\dots a_N} f_{a_1} f_{a_2}^\dagger \cdots f_{a_N}^\dagger, & D_- &= D_+^\dagger \\ R_+ &= \frac{1}{N!}\varepsilon_{a_1\dots a_N} b_{a_1}^\dagger b_{a_2} f_{a_3}^\dagger \cdots f_{a_N}^\dagger, & R_- &= R_+^\dagger \\ X_+ &= \frac{1}{N!}\varepsilon_{a_1\dots a_N} b_{a_1}^\dagger f_{a_2}^\dagger \cdots f_{a_N}^\dagger, & Y_- &= X_+^\dagger \\ X_- &= \frac{1}{N!}\varepsilon_{a_1\dots a_N} b_{a_1} f_{a_2}^\dagger \cdots f_{a_N}^\dagger, & Y_+ &= X_-^\dagger. \end{aligned} \tag{C25}$$

In the representation $\bar{R}$ we have

$$\begin{aligned} \bar{C}_- &= \frac{1}{N!}\varepsilon_{a_1\dots a_N} \bar{f}_{a_1}^\dagger \cdots \bar{f}_{a_N}^\dagger, & \bar{C}_+ &= \bar{C}_-^\dagger \\ \bar{D}_- &= \frac{1}{N!}\varepsilon_{a_1\dots a_N} \bar{f}_{a_1} \bar{f}_{a_2}^\dagger \cdots \bar{f}_{a_N}^\dagger, & \bar{D}_+ &= \bar{D}_-^\dagger \\ \bar{R}_- &= \frac{1}{N!}\varepsilon_{a_1\dots a_N} \bar{b}_{a_1}^\dagger \bar{b}_{a_2} \bar{f}_{a_3}^\dagger \cdots \bar{f}_{a_N}^\dagger, & \bar{R}_+ &= \bar{R}_-^\dagger \\ \bar{Y}_- &= \frac{1}{N!}\varepsilon_{a_1\dots a_N} \bar{b}_{a_1}^\dagger \bar{f}_{a_2}^\dagger \cdots \bar{f}_{a_N}^\dagger, & \bar{X}_+ &= \bar{Y}_-^\dagger \\ \bar{Y}_+ &= \frac{1}{N!}\varepsilon_{a_1\dots a_N} \bar{b}_{a_1} \bar{f}_{a_2}^\dagger \cdots \bar{f}_{a_N}^\dagger, & \bar{X}_- &= \bar{Y}_+^\dagger. \end{aligned} \tag{C26}$$

Then H_1 projected onto $R \otimes \bar{R}$ is proportional to the osp(2|2) invariant product of these operators, given by

$$\begin{aligned} H_1 = \frac{N}{2}\big[& C_+\bar{C}_- + C_-\bar{C}_+ - N(D_+\bar{D}_- + D_-\bar{D}_+) \\ & - N(N-1)(R_+\bar{R}_- + R_-\bar{R}_+) \\ & - N(X_+\bar{Y}_- - Y_-\bar{X}_+ + X_-\bar{Y}_+ - Y_+\bar{X}_-)\big]. \end{aligned} \tag{C27}$$

The terms of H_1 that have non-vanishing matrix elements within $R \otimes \bar{R}$ flip the states between the subspaces $R_0 \otimes \bar{R}_0$ and $R_N \otimes \bar{R}_N$.

The term

$$H_\omega = 2\omega(B + Q_3 - \bar{B} - \bar{Q}_3) \tag{C28}$$

breaks the supersymmetry from osp(2|2) down to a u(1|1) subalgebra, which is generated by

$$Q_3, B, V_-, W_+. \tag{C29}$$

3. Eigenvalue problem

For non-zero ω in H_ω the symmetry of the Hamiltonian H_r in Eq. (C12) is u(1|1) of Eq. (C29). We can use this to find eigenstates of H_r in the singlet sector of this u(1|1). Basis states in this sectors $|m\rangle_i$ are defined in terms of the states in $R \otimes \bar{R}$ as follows:

$$\begin{aligned}
|0\rangle_1 &= |0\rangle|\overline{0}\rangle, \\
|m\rangle_1 &= \frac{1}{\sqrt{2}}\Big(\big|2m,0\big\rangle\big|\overline{2m,0}\big\rangle \\
&\quad + \big|2m-1,1\big\rangle\big|\overline{2m-1,1}\big\rangle\Big), && m \geqslant 1, \\
|m\rangle_2 &= \frac{1}{\sqrt{2}}\Big(\big|2m-2,N\big\rangle\big|\overline{2m-2,N}\big\rangle \\
&\quad + \big|2m-1,N-1\big\rangle\big|\overline{2m-1,N-1}\big\rangle\Big), && m \geqslant 1.
\end{aligned} \tag{C30}$$

We look for eigenstates of H of the form

$$|\psi\rangle = \psi_0|0\rangle_1 + \sum_{m=1}^{\infty}(\psi_m^{(1)}|m\rangle_1 + \psi_m^{(2)}|m\rangle_2), \tag{C31}$$

where the coefficients ψ_0, $\psi_m^{(1)}$, and $\psi_m^{(2)}$ satisfy some recursion relations, and can be found from them. The relations are obtained by plugging the ansatz (C31) into the eigenvalue problem $H_r|\psi\rangle = E_r|\psi\rangle$ and equating the coefficients of basis states on both sides of the equation.

First, using the definitions (C13, C14, C25, C26) of the operators appearing in the Hamiltonians (C24, C27, C28), we obtain the action of the parts of the Hamiltonian H_r.

$$\begin{aligned}
H_0|0\rangle_1 &= -n\sqrt{2}|1\rangle_1, \\
H_0|m\rangle_1 &= -(m+1)(m+n)|m+1\rangle_1 \\
&\quad + m(2m+2n-1)|m\rangle_1 \\
&\quad - (m-1)(m+n-1)|m-1\rangle_1, && m \geqslant 1, \\
H_0|m\rangle_2 &= -m(m+n)|m+1\rangle_2 \\
&\quad + (2m-1)(m+n-1)|m\rangle_2 \\
&\quad - (m-1)(m+n-2)|m-1\rangle_2, && m \geqslant 1.
\end{aligned} \tag{C32}$$

$$\begin{aligned}
H_1|0\rangle_1 &= n\sqrt{2}|1\rangle_2, \\
H_1|m\rangle_1 &= (m+n)|m+1\rangle_2 \\
&\quad - (m+n-1)|m\rangle_2, && m \geqslant 1, \\
H_1|m\rangle_2 &= m|m\rangle_1 - (m-1)|m-1\rangle_1, && m \geqslant 1.
\end{aligned} \tag{C33}$$

$$\begin{aligned}
H_\omega|m\rangle_1 &= 2m\omega|m\rangle_1, \\
H_\omega|m\rangle_2 &= 2(m+n-1)\omega|m\rangle_2, && \forall m.
\end{aligned} \tag{C34}$$

Note, that in the subspace spanned by the states $|m\rangle_i$ we have

$$H_1^2|m\rangle_i = -H_0|m\rangle_i. \tag{C35}$$

(It is to have the simplest possible coefficient in this equation that the constant λ was introduced as in Eq. (C10).) This relation will significantly simplify the analysis of the recursion relations for $\psi_m^{(i)}$ for small m (see section C 5).

Combining the above equations, we get the following recursion relations:

$$\begin{aligned}
E_r\psi_0 &= 0, \\
(2\omega - E - R)\psi_1^{(1)} &= (n+1)\psi_2^{(1)} - (2n+1)\psi_1^{(1)} + n\sqrt{2}\psi_0 + 2\lambda(\psi_2^{(2)} - \psi_1^{(2)}), \\
(2n\omega - E_r)\psi_1^{(2)} &= n\big(\psi_2^{(2)} - \psi_1^{(2)} + 2\lambda(\psi_1^{(1)} - \sqrt{2}\psi_0)\big), \\
-E_r(\Delta\psi_m^{(1)} + \psi_m^{(1)}) &= (m+1)\left((m+n+1)\Delta^2\psi_m^{(1)} + (1-2\omega)\Delta\psi_m^{(1)} - 2\omega\psi_m^{(1)} + 2\lambda(\Delta^2\psi_m^{(2)} + \Delta\psi_m^{(2)})\right), && m \geqslant 2, \\
-E_r(\Delta\psi_m^{(2)} + \psi_m^{(2)}) &= (m+n)\left((m+1)\Delta^2\psi_m^{(2)} + (1-2\omega)\Delta\psi_m^{(2)} - 2\omega\psi_m^{(2)} + 2\lambda\Delta\psi_m^{(1)}\right), && m \geqslant 2.
\end{aligned} \tag{C36}$$

The last two equations are written in terms of the finite differences:

$$\begin{aligned}
\Delta\psi_m &= \psi_{m+1} - \psi_m, \\
\Delta^2\psi_m &= \psi_{m+2} - 2\psi_{m+1} + \psi_m,
\end{aligned} \tag{C37}$$

Our strategy for the analysis of these equations is the following. For large $m \gg n$ we can treat m as a continuous variable. The solutions of the resulting equation are required to decay at infinity. The solutions of the continuous equation show that the effect of small ω becomes sig-

nificant only for $m \gtrsim \omega^{-1}$. Then for $m \ll \omega^{-1}$ we can neglect ω in the equation (C36). The corresponding Hamiltonian (within the space $R \otimes \bar{R}$) is simply $2\lambda H_1 - H_1^2$. Then, instead of solving Eq. (C36), we can first find the eigenstates of H_1 and then combine them to obtain the energy eigenstates.

After finding approximate solutions in two regions $m \gg n$ and $m \ll \omega^{-1}$, we can match them in the overlapping region $n \ll m \ll \omega^{-1}$ (we choose ω to be asymptotically small so that this region is large). For the ground state this matching procedure gives a complete solution. We can use this solution to calculate the expectation value of the "order parameter" in the ground state, which gives the density of states in the 1D wire.

For excited states the matching procedure gives the spectrum of energies E of the coupled superspins. Then the localization length $\xi_{\rm mean}$ associated with average correlators in the state at the Fermi energy in the original 1D wire problem is given by the inverse of the gap in the energy spectrum above the ground state in the superspin problem. Let us implement this strategy step by step.

4. Solution of Eqs. (C36) for large m

For large $m \gg n$, m can be treated as a continuous variable, and we can replace finite differences by derivatives with respect to m:

$$\psi_m^{(i)} \to \psi_i(m), \quad \Delta\psi_m^{(i)} \to \psi_i'(m), \quad \Delta^2\psi_m^{(i)} \to \psi_i''(m). \tag{C38}$$

In this limit we neglect ω and constants of order 1 (including n) compared to m. Assuming a smooth variation of $\psi_i(m)$ we also neglect $\psi_i''(m)$ compared to $\psi_i'(m)$, and $\psi_i'(m)$ compared to $\psi_i(m)$. Then the Eqs. (C36) become

$$\begin{aligned}(2\omega m - E_r)\psi_1(m) &= m\big(m\psi_1''(m) + \psi_1'(m) + 2\lambda\psi_2'(m)\big),\\ (2\omega m - E_r)\psi_2(m) &= m\big(m\psi_2''(m) + \psi_2'(m) + 2\lambda\psi_1'(m)\big).\end{aligned} \tag{C39}$$

To solve the equtions (C39), we introduce

$$\psi_\pm(m) = \psi_1(m) \pm \psi_2(m). \tag{C40}$$

Then, Eqs. (C39) can be written as

$$(2\omega m - E_r)\psi_\pm(m) = m^2\psi_\pm''(m) + (1 \pm 2\lambda)m\psi_\pm'(m). \tag{C41}$$

After a change of variable $x^2 = 8\omega m$, and the substitution $\psi_\pm(x) = x^{\mp 2\lambda} f_\pm(x)$, we get the modified Bessel equation

$$x^2 f_\pm''(x) + x f_\pm'(x) - \big(x^2 + 4(\lambda^2 - E_r)\big) f_\pm(x) = 0. \tag{C42}$$

The decaying solutions for $\psi_\pm$ can now be written as

$$\psi_\pm(x) = 2A_\pm \left(\frac{x}{2}\right)^{\mp 2\lambda} K_{2\mu}(x), \tag{C43}$$

where $A_\pm$ are constants, K the modified Bessel function, and

$$\mu = \sqrt{\lambda^2 - E_r}. \tag{C44}$$

For the purpose of matching these solutions to the ones for $m \ll \omega^{-1}$ we need the small x behavior of the Bessel function $K_\mu(x)$. To obtain it we use the definition

$$K_\mu(x) = \frac{\pi}{2\sin\mu\pi}\big(I_{-\mu}(x) - I_\mu(x)\big) \tag{C45}$$

and the small x asymptotics for $I_\mu(x)$:

$$I_\mu(x) \sim \frac{(x/2)^\mu}{\Gamma(1+\mu)}. \tag{C46}$$

Then we get for small x

$$K_\mu(x) \sim \frac{1}{2}\left(\Gamma(\mu)\left(\frac{x}{2}\right)^{-\mu} + \Gamma(-\mu)\left(\frac{x}{2}\right)^{\mu}\right). \tag{C47}$$

After the substitution $x = 2(2\omega m)^{1/2}$ the limiting form of $\psi_\pm$ for small m becomes

$$\psi_\pm(m) \sim A_\pm\left[\Gamma(2\mu)(2\omega m)^{-\mu\mp\lambda} + \Gamma(-2\mu)(2m\omega)^{\mu\mp\lambda}\right]. \tag{C48}$$

For the ground state $E_r = 0$, we have $\mu = |\lambda|$, and the asymptotic solution (C48) takes the form

$$\psi_\pm(m) \sim A_\pm\left[\Gamma(\mp 2\lambda) + \Gamma(\pm 2\lambda)(2m\omega)^{\mp 2\lambda}\right]. \tag{C49}$$

Let us specify the previous solutions to the case of $\lambda = 0$. Eq. (C48) for an excited state wave function simply becomes

$$\psi_\pm(m) \sim A_\pm\left[\Gamma(2\mu)(2\omega m)^{-\mu} + \Gamma(-2\mu)(2\omega m)^{\mu}\right], \tag{C50}$$

where now $\mu = i\sqrt{E_r}$. For the ground state we have from Eq. (C43)

$$\psi_\pm(m) = 2A_\pm K_0(x) \sim -A_\pm\big(\ln(2\omega m) + 2\gamma\big), \tag{C51}$$

where γ is the Euler's constant.

5. Solution of Eqs. (C36) for small m

In the limit $m \ll \omega^{-1}$, H_ω can be neglected and thus we have

$$\left[2\lambda H_1 - H_1^2\right]|\psi\rangle = E_r|\psi\rangle, \tag{C52}$$

where we used Eq. (C35). Given the eigenstates of H_1:

$$H_1|\psi, j\rangle = j|\psi, j\rangle, \tag{C53}$$

the eigenvalues j are related to E_r by

$$j^2 - 2\lambda j + E_r = 0, \tag{C54}$$

which has two solutions

$$j_{1,2} = \lambda \pm \mu, \tag{C55}$$

with the same μ as before, see Eq. (C44). The energy eigenstates are then given by

$$|\psi\rangle = A_1\big|\psi, j_1\big\rangle + A_2\big|\psi, j_2\big\rangle. \tag{C56}$$

Equation (C53) leads to the recursion relations

$$\begin{aligned} &j\psi_0 = 0, \\ &j\psi_1^{(2)} = n(\sqrt{2}\psi_0 - \psi_1^{(1)}), \\ &j\psi_m^{(1)} = m(\psi_m^{(2)} - \psi_{m+1}^{(2)}), && m \geqslant 1, \\ &j\psi_m^{(2)} = (m+n-1)(\psi_{m-1}^{(1)} - \psi_m^{(1)}). && m \geqslant 2. \end{aligned} \tag{C57}$$

First, we consider excited states with $E_r > 0$. Then $j \neq 0$, and it follows from the first of Eqs. (C57) that

$$\psi_0 = 0. \tag{C58}$$

The rest of Eqs. (C57) can be solved using the generating functions

$$f_i(z) = \sum_{m=1}^{\infty} \psi_m^{(i)} z^{m-1}. \tag{C59}$$

The coefficients $\psi_m^{(i)}$ are obtained from the generating functions as

$$\psi_m^{(i)} = \oint \frac{dz}{2\pi i}\, z^{-m} f_i(z), \tag{C60}$$

where the integral is taken counterclockwise around zero.

From Eqs. (C57), we obtain the equations that the generating functions satisfy:

$$\begin{aligned} z(1-z)\frac{df_1}{dz} + [n - (1+n)z]\, f_1 + j f_2 = 0 \\ (1-z)\frac{df_2}{dz} - f_2 + j f_1 = 0. \end{aligned} \tag{C61}$$

Substitution of $f_1(z)$ from the second Eq. (C61) into the first one, and the change $f_2(z) = (1-z)^{j-1}u(z)$, lead to a hypergeometric equation

$$z(1-z)\frac{d^2u}{dz^2} + \big(n - (n+2j+1)z\big)\frac{du}{dz} - j(j+n)u = 0, \tag{C62}$$

with the solution which is regular near $z = 0$

$$u(z) = F(j, j+n; n; z). \tag{C63}$$

Thus, in terms of the hypergeometric function

$$\psi_m^{(2)} = A \oint \frac{dz}{2\pi i}\, z^{-m}(1-z)^{j-1} F(j, j+n; n; z), \tag{C64}$$

where the constant A is determined later by matching this solution to the one for large m.

To evaluate the last integral, we use one of the Kummer's relations for the hypergeometric function:

$$(1-z)^{j-1}F(j, j+n; n; z) = \frac{\Gamma(n)\Gamma(-2j)}{\Gamma(n-j)\Gamma(-j)}(1-z)^{j-1}z^{-j}F(j, 1-n+j; 1+2j; 1-z^{-1}) + (j \leftrightarrow -j). \tag{C65}$$

Then

$$\psi_m^{(2)} = A\left(\frac{\Gamma(n)\Gamma(-2j)}{\Gamma(n-j)\Gamma(-j)} I(j) + (j \leftrightarrow -j)\right), \tag{C66}$$

$$I(j) = \oint \frac{dz}{2\pi i}\, z^{-(m+j)}(1-z)^{j-1} F(j, 1-n+j; 1+2j; 1-z^{-1}). \tag{C67}$$

In the last integral we deform the contour to run along the branch cut $[1, \infty)$. On the upper and lower sides of the cut $1 - z = e^{-i\pi}(z-1)$ and $1 - z = e^{i\pi}(z-1)$, correspondingly. Then the sum of the contributions from the two sides is

$$I(j) = \frac{\sin \pi j}{\pi} \int_1^{\infty} dz\, z^{-(m+j)}(z-1)^{j-1} F(j, 1-n+j; 1+2j; 1-z^{-1}). \tag{C68}$$

The change of variables $x = 1 - z^{-1}$ leads to a tabulated integral:

$$\int_0^1 dx\, x^{j-1}(1-x)^{m-1} F(j, 1-n+j; 1+2j; x) = \frac{\Gamma(j)\Gamma(m)}{\Gamma(m+j)}\, {}_3F_2(j, 1-n+j, j; 1+2j, m+j; 1), \tag{C69}$$

see Eq. 7.512.5 in Gradshteyn and Ryzhik[38]. This provides us with the exact solution of the recursion relations

(C57). However, we really need only the asymptotic form of this solution for large m:

$$I(j) = \frac{1}{\Gamma(1-j)} m^{-j}(1 + O(1/m)). \tag{C70}$$

Denoting

$$\Lambda(j) = \frac{\Gamma(n)\Gamma(2j)}{\Gamma(n+j)\Gamma(j)\Gamma(1+j)}, \tag{C71}$$

we have the asymptotic behavior of $\psi_m^{(2)}$ for large m:

$$\psi_m^{(2)} \sim A\big(\Lambda(j)m^j + \Lambda(-j)m^{-j}\big). \tag{C72}$$

Next we substitute this asymptotic form into the recursion relation for $\psi_m^{(1)}$ in Eq. (C57), and obtain with the same accuracy

$$\psi_m^{(1)} \sim -A\big(\Lambda(j)m^j - \Lambda(-j)m^{-j}\big). \tag{C73}$$

For a given j we get the asymptotics for the functions $\psi_\pm$

$$\psi_\pm(m) \sim \pm 2A\Lambda(\mp j)m^{\mp j}. \tag{C74}$$

Combining these for both values of j, we get, finally, for excited states

$$\psi_\pm(m) \sim \pm 2\big(A_1\Lambda(\mp j_1)m^{\mp j_1} + A_2\Lambda(\mp j_2)m^{\mp j_2}\big). \tag{C75}$$

Next, we turn to the ground state. In this case $E_r = 0$, and we have

$$j_1 = 2\lambda, \qquad j_2 = 0. \tag{C76}$$

For $j = 0$ the Eqs. (C57) give

$$\begin{aligned} \psi_0 &= \text{const}, \\ \psi_m^{(1)} &= \psi_1^{(1)} = \sqrt{2}\psi_0, \\ \psi_m^{(2)} &= \psi_1^{(2)}. \end{aligned} \tag{C77}$$

Then

$$\psi_\pm(m) = \sqrt{2}\psi_0 \pm \psi_1^{(2)}. \tag{C78}$$

Combining this with Eq. (C74) we find that the in the case $\lambda \neq 0$ the Eq. (C75) specializes to

$$\psi_\pm(m) \sim \pm 2A_1\Lambda(\mp 2\lambda)m^{\mp 2\lambda} + \sqrt{2}\psi_0 \pm \psi_1^{(2)}. \tag{C79}$$

In the special case of $\lambda = 0$ the roots j_1 and j_2 become degenerate, and we have to be more careful. In this case it is actually easier to come back to the recursion relations (C36). The first of these leaves ψ_0 an unspecified constant, while the rest can now be written as

$$\begin{aligned} (n+1)\big(\psi_2^{(1)} - \psi_1^{(1)}\big) &= n\big(\psi_1^{(1)} - \sqrt{2}\psi_0\big), \\ \psi_2^{(2)} - \psi_1^{(2)} &= 0, \\ \Delta\left[(m+n)\Delta\psi_m^{(1)}\right] &= 0, \qquad m \geqslant 2, \\ \Delta\left[m\Delta\psi_m^{(2)}\right] &= 0, \qquad m \geqslant 2. \end{aligned} \tag{C80}$$

Integrating the last pair of equations (C80), and finding integration constants from the first pair, we obtain

$$\begin{aligned} (m+n)\left[\psi_{m+1}^{(1)} - \psi_m^{(1)}\right] &= n\big(\psi_1^{(1)} - \sqrt{2}\psi_0\big), \\ m\left[\psi_{m+1}^{(2)} - \psi_m^{(2)}\right] &= 0. \end{aligned} \tag{C81}$$

Thus,

$$\begin{aligned} \psi_m^{(1)} &= \psi_{m-1}^{(1)} + \frac{n\big(\psi_1^{(1)} - \sqrt{2}\psi_0\big)}{n+m-1} \\ &= \sqrt{2}\psi_0 + n\big(\psi_1^{(1)} - \sqrt{2}\psi_0\big)\left[\Psi(n+m) - \Psi(n)\right], \end{aligned} \tag{C82}$$

where Ψ is the Digamma function. For $\psi_m^{(2)}$ on the other hand, we have

$$\psi_m^{(2)} = \psi_1^{(2)}. \tag{C83}$$

For the combinations $\psi_\pm$ this gives

$$\begin{aligned} \psi_\pm(m) &= \sqrt{2}\psi_0 \pm \psi_1^{(2)} \\ &\quad + n\big(\psi_1^{(1)} - \sqrt{2}\psi_0\big)\left[\Psi(n+m) - \Psi(n)\right], \end{aligned} \tag{C84}$$

with the limiting form, for large m,

$$\begin{aligned} \psi_\pm(m) &\sim \sqrt{2}\psi_0 \pm \psi_1^{(2)} \\ &\quad + n\big(\psi_1^{(1)} - \sqrt{2}\psi_0\big)\left[\ln m - \Psi(n)\right]. \end{aligned} \tag{C85}$$

6. Matching solutions for large and small m

Let us start with the ground state $E_r = 0$ and a generic $\lambda \neq 0$. In this case we need to match the Eqs. (C49) and (C79). This gives the conditions

$$\begin{aligned} A_\pm\Gamma(\mp 2\lambda) &= \sqrt{2}\psi_0 \pm \psi_1^{(2)}, \\ A_\pm\Gamma(\pm 2\lambda)(2\omega)^{\mp 2\lambda} &= \pm 2A_1\Lambda(\mp 2\lambda). \end{aligned} \tag{C86}$$

Solving for A_+ and A_-, we get

$$\begin{aligned} A_- &= \frac{2\sqrt{2}\psi_0}{\Gamma(2\lambda)\big(1 - \Phi(\lambda)(2\omega)^{4\lambda}\big)}, \\ A_+ &= -\frac{2\sqrt{2}\psi_0}{\Gamma(-2\lambda)} \frac{\Phi(\lambda)(2\omega)^{4\lambda}}{1 - \Phi(\lambda)(2\omega)^{4\lambda}}, \end{aligned} \tag{C87}$$

where

$$\Phi(\lambda) = \frac{\Gamma^2(-2\lambda)\Lambda(-2\lambda)}{\Gamma^2(2\lambda)\Lambda(2\lambda)}. \tag{C88}$$

For finite λ the leading behavior for asymptotically small ω is then given by

$$\begin{aligned} &A_+ \propto \omega^{4\lambda}, \qquad A_- = \text{const}, \qquad \text{if } \lambda > 0, \\ &A_+ = \text{const}, \qquad A_- \propto \omega^{4|\lambda|}, \qquad \text{if } \lambda < 0. \end{aligned} \tag{C89}$$

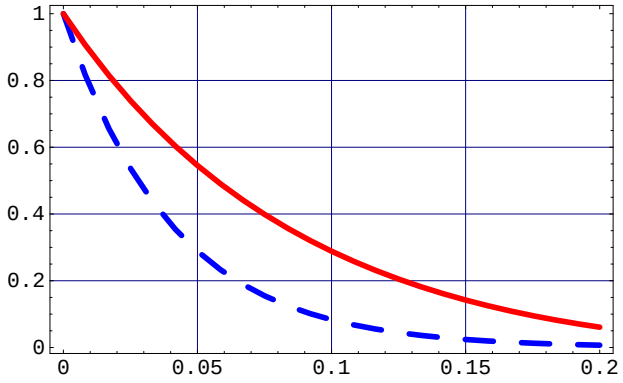

FIG. 3: Solid and dashed lines show the right and left hand sides of Eq. (C96) for $n = 5$, $\lambda = 0.2$, and $\omega = 0.001$.

For $\lambda = 0$, matching Eq. (C85) and Eq. (C51) gives

$$\psi_m^{(2)} = 0, \qquad \forall m,$$
$$A_\pm = \frac{-\sqrt{2}\psi_0}{\ln(C\omega)}, \qquad C = 2e^{2\gamma+\Psi(n)}. \tag{C90}$$

Essentially the same result is obtained by taking the limit of $\lambda \to 0$ in Eqs. (C87). Indeed, in this limit

$$\Gamma(2\lambda) \to \frac{1}{2\lambda}, \qquad \Phi(\lambda) \to 1,$$
$$\lim_{\lambda\to 0} \frac{1-(2\omega)^{4\lambda}}{2\lambda} = -2\ln(2\omega), \tag{C91}$$

and Eq. (C90) follows (but with $C = 2$ now).

The final form of the ground state (with the choice of $\psi_0 = 1$) for general $\lambda \neq 0$ is

$$\psi_\pm(m) = A_\pm\Big(\Gamma(\mp 2\lambda) + \Gamma(\pm 2\lambda)(2m\omega)^{\mp 2\lambda}\Big), \quad m \ll \frac{1}{\omega},$$
$$= 2A_\pm(2\omega m)^{\mp\lambda} K_{2\lambda}(2\sqrt{2\omega m}), \qquad m \gg n. \tag{C92}$$

For $\lambda = 0$ we have, instead,

$$\psi_\pm(m) = \sqrt{2}\left(1 + \frac{\Psi(m+n) - \Psi(n)}{\ln(C\omega)}\right), \quad m \ll \frac{1}{\omega}$$
$$= -\frac{2\sqrt{2}}{\ln(C\omega)} K_0\big(2\sqrt{2\omega m}\big), \qquad m \gg n. \tag{C93}$$

Next we discuss excited states with

$$E_r = \lambda^2 - \mu^2, \tag{C94}$$

see Eq. (C44). The value of μ may be obtained by matching the large m and small m limits of the ψ^i's in the excited states, Eqs. (C48) and (C75) and by requiring a consistent, non-trivial solution. Unlike in the ground state, we now have $\psi_0 = 0$, and thus the system of equations for coefficients $A_\pm, A_{1,2}$ obtained from the matching is homogenous:

$$A_+\Gamma(2\mu)(2\omega)^{-\mu-\lambda} = 2A_1\Lambda(-\mu-\lambda),$$
$$A_+\Gamma(-2\mu)(2\omega)^{\mu-\lambda} = 2A_2\Lambda(\mu-\lambda),$$
$$A_-\Gamma(2\mu)(2\omega)^{-\mu+\lambda} = -2A_2\Lambda(-\mu+\lambda),$$
$$A_-\Gamma(2\mu)(2\omega)^{\mu+\lambda} = -2A_1\Lambda(\mu+\lambda). \tag{C95}$$

A non-trivial solution of this system exists only when the corresponding determinant vanishes, and this gives

$$(2\omega)^{4\mu} = \frac{\Gamma^2(2\mu)\Lambda(\mu-\lambda)\Lambda(\mu+\lambda)}{\Gamma^2(-2\mu)\Lambda(-\mu-\lambda)\Lambda(-\mu+\lambda)}. \tag{C96}$$

Before analyzing this equation, let us note that, according to the definition in Eq. (C44), if μ is real, then it must lie in the range

$$0 \leqslant \mu \leqslant |\lambda|. \tag{C97}$$

It can also take purely imaginary values.

A quick look at the right hand side (RHS) of the Eq. (C96) shows that the value $|\lambda| = 1/4$ is special, and we will restrict ourselves to the values $|\lambda| < 1/4$. In this case for the allowed real values of μ, the RHS of the Eq. (C96) monotonically decreases from 1 to

$$-\frac{\Gamma(4|\lambda|)\Gamma(n-2|\lambda|)}{\Gamma(-4|\lambda|)\Gamma(n+2|\lambda|)} > 0.$$

Then for any n there is a value

$$\omega_n = \frac{1}{2}\left(-\frac{\Gamma(4|\lambda|)\Gamma(n-2|\lambda|)}{\Gamma(-4|\lambda|)\Gamma(n+2|\lambda|)}\right)^{1/4|\lambda|},$$

such that for $\omega < \omega_n$ there are no real solution for μ, apart from $\mu = 0$. A typical situation is shown in Fig. 3.

The absence of real solutions for μ means that the energy spectrum of the reduced Hamiltonian H_r has a gap λ^2 above the ground state energy (which is 0 by supersymmetry). This, in turn, implies that the mean localization length in the original disordered wire problem is finite

$$\xi_{\text{mean}} = 2\left(\frac{2c_2}{a}\lambda^2\right)^{-1}, \tag{C98}$$

except at $\lambda = 0$, where it diverges. After substitution of λ from Eq. (C10), and $c_2 = a/(\gamma\ell)$, the last expression reduces exactly to the one given by Eqs. (42–44, 46).

For purely imaginary μ, however, Eq. (C96) has infinitely many solutions, since $(2\omega)^{2\mu} = (2\omega)^{2\mu}e^{2\pi i l}$, for any integer l. Upon expanding both sides of Eq. (C96) for small $|\mu|$, one finds that in the limit of $\omega \ll 1$, μ takes on a discrete set of values,

$$\mu \approx i\frac{\pi l}{2|\ln(2\omega)|}, \quad l = 0, 1, 2, \ldots, \tag{C99}$$

which gives for the energy eigenvalues

$$E_r = \lambda^2 + \frac{\pi^2 l^2}{4\ln^2(2\omega)}. \tag{C100}$$

For $\omega \to 0$, the energy levels above the gap merge into a continuum.

7. Density of states

Expressions (C92,C93) for the ground state wave function allow us to determine the leading behavior of the average density of states $\nu(\epsilon)$ in the wire for asymptotically low energies ϵ. The density of states (DOS) $\nu(\epsilon)$ for the original problem of a 1D superconductor is related to the "order parameter" of the superspin problem. The "order parameter" is the measure of the amount of the symmetry breaking induced by ω in the ground state, and is expressed as the ground state expectation value of, say, the staggered fermionic number

$$\hat{O} = \bar{B} - B = \frac{1}{2}(2n - n_f - n_{\bar{f}}).$$

The density of states is found by analytical continuation $\omega \to i\epsilon$ as follows:

$$\nu(\epsilon) \propto \mathrm{Re}\Big(\langle \hat{O}\rangle_0\big|_{\omega\to i\epsilon}\Big), \tag{C101}$$

Given the form of the singlets in Eq. (C30), we have

$$\begin{aligned}
\hat{O}|0\rangle_1 &= n|0\rangle_1,\\
\hat{O}|m\rangle_1 &= \frac{1}{\sqrt{2}}\Big(n\big|2m,0\big\rangle\big|\overline{2m,0}\big\rangle\\
&\quad + (n-1)\big|2m-1,1\big\rangle\big|\overline{2m-1,1}\big\rangle\Big),\\
\hat{O}|m\rangle_2 &= \frac{1}{\sqrt{2}}\Big((-n\big|2m-2,N\big\rangle\big|\overline{2m-2,N}\big\rangle\\
&\quad - (n-1)\big|2m-1,N-1\big\rangle\big|\overline{2m-1,N-1}\big\rangle\Big).
\end{aligned} \tag{C102}$$

Using the fact that the states $\big|\overline{2m-1,1}\big\rangle$ and $\big|\overline{2m-1,N-1}\big\rangle$ have negative square norms, we obtain

$$\begin{aligned}
\langle \hat{O}\rangle_0 &= n\psi_0^2 + \frac{1}{2}\sum_{m=1}^{\infty}\Big[\big(\psi_m^{(1)}\big)^2 - \big(\psi_m^{(2)}\big)^2\Big]\\
&= n\psi_0^2 + \frac{1}{2}\sum_m \psi_+(m)\psi_-(m)
\end{aligned} \tag{C103}$$

Consider now the case $\lambda = 0$. The sum in Eq. (C103) can be broken up into two parts corresponding to the two regimes, separated by some m_0 such that $n \ll m_0 \ll 1/\omega$. The sum over $m \leqslant m_0$ gives a finite contribution as $\omega \to 0$. The rest of the sum in Eq. (C103) dominates for small ω, and can be approximated by the integral

$$\begin{aligned}
&A_+A_-\int_{m_0}^{\infty} dm\, K_0^2(2\sqrt{2m\omega})\\
&= \frac{2\psi_0^2}{\omega\ln^2(C\omega)}\int_{x_0}^{\infty} dx\, x\, K_0^2(x) \propto \frac{1}{\omega\ln^2(C\omega)}.
\end{aligned} \tag{C104}$$

Thus, for asymptotically small energy the density of states behaves as

$$\nu(\epsilon) \sim \mathrm{Re}\frac{1}{i\epsilon\ln^2(Ci\epsilon)} \sim \frac{1}{\epsilon|\ln^3\epsilon|}, \tag{C105}$$

which is precisely the Dyson's singularity of Eq. (2).

However, in the case of $\lambda \neq 0$, we have similarly to Eq. (C104) that the contribution from $m > m_0$ to the "order parameter" is

$$\langle \hat{O}\rangle_0 \sim A_+A_-\int_{m_0}^{\infty} dm\, K_{2\lambda}^2(2\sqrt{2\omega m}) \sim \omega^{4|\lambda|-1}. \tag{C106}$$

This contribution dominates the finite contribution from $m < m_0$ only if $|\lambda| \leqslant 1/4$ (for larger values of $|\lambda|$ our method is not precise enough to determine the DOS). In this range we then get the following asymptotic form for the DOS:

$$\nu(\epsilon) \sim \epsilon^{4|\lambda|-1}. \tag{C107}$$

This is precisely the power-law with a non-universal exponent depending on the closeness to criticality, which was found for a generic 1D system in classes BD and DIII by Motrunich *et. al*[7] in a real space RG study.

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